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CONTINUITY OF MULTIFUNCTIONS VIA e-OPEN SETS

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Abstract. In this paper, we introduce a stronger notion of continuity for multifunctions via e-open sets. We establish several characterizations of this new continuity concept and examine its fundamental properties. Furthermore, we investigate the preservation of certain topological properties under this newly introduced class of functions, thereby providing deeper insight into its structural significance in the context of general topology.

1. introduction

The significance of multifunctions, or set-valued functions, has been recognized since the early twentieth century, with fundamental contributions from mathematicians such as Hausdorff [12], Vietoris [32, 33], Hahn [11], and Kuratowski [17, 18]. Although various results were obtained in this period, systematic investigations began to emerge in the 1960s, motivated by applications in control theory and mathematical economics. During this time, extensive research was devoted to notions such as continuity, differentiability, measurability, integrability, homotopicity, and fixed-point theory for multifunctions (see [3, 5, 8, 9, 14, 15, 19, 28, 29]).

Today, multifunctions and multi-valued analysis have become well-established areas of mathematics in their own right. Moreover, they serve as powerful tools for addressing problems arising across diverse fields, including nonlinear analysis, nonlinear programming, mathematical economics, optimal control theory, biology, artificial intelligence, and numerous other disciplines (see [7, 13]).

The notion of continuity for multifunctions, which is the focus of this study, was first introduced by Wallace [30] in 1941. Shortly thereafter, Eilenberg and Montgomery [9] established a fixed-point theorem for multifunctions. In the following years, various definitions of continuity were proposed by different researchers (e.g. [6, 31]). Strother [28] and Lechicki [19] later examined the relationships among these definitions and investigated their equivalence. The formulation of continuity for multifunctions most commonly used today was provided by Smithson [27] using Berge's [4] notations.

After this new formulation of continuity for multifunctions, many continuity concepts originally studied for single-valued functions have since been extended to the

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setting of multifunctions. Researchers have explored the connections between multifunctions satisfying these generalized notions of continuity and traditionally continuous multifunctions, with particular attention given to the topological properties preserved by these new classes (e.g. [16, 21, 23, 24, 25, 36, 38]). At present, the study of multifunctions and their applications continues to be an active and expanding area of research (e.g. [7, 10, 22, 37]).

The concepts of e-open sets in topological spaces were introduced and considered by Afrooz et al. [1, 2]. In the first study, the properties of e-open sets as well as rings of real-valued e-continuous functions on a topological space were investigated. In the second study, the properties of the topological space defined by e-open sets were investigated and this space was compared with extremally disconnected topological spaces and zero-dimensional topological spaces.

In this study, we introduce a new strong form of continuity of multifunctions, called e-continuity by using e-open sets. We give some characterizations for this new kind of continuity and examine the properties of this new class of multifunctions. In addition, we focused on the topological properties that this new class of functions preserves.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated.

A set G in a topological space X is called extremely open (briefly e-open) [1] if G and closure of G ($Cl_X G$ or simply ClG) are open subsets of X. For all the following concepts with e-open sets we refer to [1, 2]. A subset of a topological space is called an e-closed if its complement is an e-open. Equivalently, a set is e-closed if and only if it is closed and its interior is also closed. Since the intersection of two e-open sets in a topological space (X, τ) is an e-open set, the set of all e-open subsets of X form a base for a topology τ_e on X. The space (X, τ_e) is denoted by X_e, for simplicity.

Let A be a subset of a topological space X. An element $x \in X$ is called an e-cluster point of A if each e-open subset of X containing x meets A. The set of all e-cluster points of A is called the e-closure of A and denoted by $eClA$. In fact, a point $x \in X$ is an e-cluster point of A as a subset of X if and only if x is a cluster point of A as a subset of X_e and $eClA = Cl_{X_e} A$.

Any open subset of X which is a union of e-open sets, is called E-open set, and similarly any closed subset of X is called an E-closed set if it is an intersection of e-closed subsets of X. Indeed, E-open and E-closed subsets of X, are in fact open and closed subsets of X_e, respectively. Therefore, the e-closure of a set A in a space X is an E-closed set and it is the intersection of all e-closed subsets of X containing A. Similarly, the e-interior of a set A, denoted by $eIntA$, is the union of all e-open subsets of X contained in A. Indeed $eIntA = Int_{X_e} A$.

By a multifunction $F : X \rightrightarrows Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction, $F : X \rightrightarrows Y$, the upper and lower inverse of any subset B of Y, denoted by $F^+(B)$ and $F^-(B)$ respectively, are the subsets $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) =$

$\{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each $y \in Y$, and the image of an $A \subseteq X$ under F is $F(A) = \cup\{F(x) : x \in A\}$. Note that $X - F^+(B) = F^-(Y - B)$ for each $B \subseteq Y$. A multifunction $F : (X, \tau) \rightrightarrows (Y, \sigma)$ is said to be upper semi continuous [4], abbreviated as u.s.c., (resp. lower semi continuous [4] or l.s.c.) at $x \in X$ if for each open $V \subseteq Y$ with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there is an open neighbourhood U of x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \emptyset$ for all $z \in U$). F is u.s.c. (resp. l.s.c.) iff it is u.s.c. (resp. l.s.c.) at each point of X. Then F is called semi continuous iff it is both u.s.c. and l.s.c.

First, let us give the following lemma, which we will use later, without proof.

Lemma 2.1. *Let X be a topological space and $A, B \subseteq X$. Then the following are hold:*

(1) $A \subseteq B \Rightarrow eClA \subseteq eClB$.

(2) $X \backslash (eClA) = eInt(X \backslash A)$.

3. e-Continuous Multifunctions

Definition 3.1. A multifunction $F : (X, \tau) \rightrightarrows (Y, \sigma)$ is said to be:

(i) upper e-continuous (briefly, u.e-c.) at a point $x \in X$ if for each open set V of Y containing $F(x)$, there exists an e-open set U in X containing x such that $F(U) \subseteq V$.

(ii) lower e-continuous (briefly, l.e-c.) at a point $x \in X$ if for each open set V of Y that satisfies $F(x) \cap V \neq \emptyset$, there exists an e-open set U in X containing x such that $F(z) \cap V \neq \emptyset$ for all $z \in U$.

(iii) upper (lower) e-continuous if F has this property at each point of X.

Remark 3.1. It is clear from the definitions that upper (resp. lower) e-continuous multifunctions are upper (resp. lower) semi continuous. The following examples show that the inverses of these requirements are not generally true.

Example 3.1. Consider the $\mathbb{R}$ with the usual topology. It is noted in [1] that an open subset G of $\mathbb{R}$ is e-open in $\mathbb{R}$ if and only if $\mathbb{R} \backslash G$ has an empty interior. Then the set valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $F(x) = \begin{cases} (-1, 1) & ; x = 0 \\ \{0\} & ; x \in [-1, 1] \backslash \{0\} \\ \{5\} & ; otherwise \end{cases}$ is both upper and lower semi continuous at $x = 0$, but neither upper nor lower e-continuous at $x = 0$.

In the following, we will investigate the equivalence of continuity and e-continuity.

Theorem 3.2. [1] *A space (X, τ) is called e-space if $\tau = \tau_e$.*

Theorem 3.3. *Let X be an e-space, $x \in X$ and $F : X \rightrightarrows Y$ be a multifunction. If F is u(l).s.c. at x, then it is u(l).e-c. at x.*

Proof. We will only prove for u.s.c. The other can be done in a similar way. Let $F : X \rightrightarrows Y$ be an u.s.c multifunction and $x \in X$. Let us take an arbitrary open set $V \subseteq Y$ satisfying the condition $F(x) \subseteq V$. Since F is u.s.c, there is an open set U with $x \in U \subseteq F^+(V)$. On the other hand, since X is an e-space, U is an e-open set. This shows that F is u.e-c. at $x \in X$. $\square$

Corollary 3.4. *Let X be an e-space. Then a multifunction $F : X \rightrightarrows Y$ is u(l).e-c. iff F is u(l).s.c.*

The following result shows that a characterization of e-spaces can be achieved with the help of e-continuity.

Corollary 3.5. *Let X be any topological space. Then the multifunction $F : X \rightrightarrows X$, $F(x) = \{x\}$ is u(l).e-c. iff X is e-space.*

Proof. ($\Rightarrow$) It is obvious.

($\Leftarrow$) Since $F : X \rightrightarrows X$, $F(x) = \{x\}$ is u(l).s.c., it is a corollary of Theorem 3.3. □

It is important to note here that, as can be seen in the example below, upper e-continuity and lower e-continuity are independent concepts in generally.

Example 3.6. Let $Y = \mathbb{R}$ with the usual topology $\tau_{\mathcal{U}}$ and let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a, b\}, \{c\}\}$. Define a multifunction $F : (X, \tau) \rightrightarrows (\mathbb{R}, \tau_{\mathcal{U}})$ by $F(a) = [1, 5]$, $F(b) = [1, 2]$, $F(c) = (7, +\infty)$. Then F is u.e-c. at $a \in X$ but not l.e-c. at $a \in X$. Additionally, F is l.e-c. at $b \in X$ but not u.e-c. at $a \in X$. Also, F is both u.e-c. and l.e-c. at $c \in X$.

Now let's look at the equivalence of upper e-continuity and lower e-continuity. Note that a multifunction $F : X \to Y$ is called image-P if $F(x)$ has the property P for every $x \in X$ (see [4, 27]). Also we recall that if $F : X \rightrightarrows Y$ is a multifunction, then $F^- : Y \rightrightarrows X$ is also a multifunction and it is clear that if F^- is image-e-open, then F is l.e-c.

Definition 3.2. [34] A multifunction $F : X \to Y$ is said to be have nonmingled point images provided that for $x_1, x_2 \in X$ with $x_1 \neq x_2$, the image sets $F(x_1)$ and $F(x_2)$ are either disjoint or identical.

It has been proven in [34] that, F is image-nonmingled iff $F \circ F^- \circ F = F$.

Theorem 3.7. *Let $F : X \to Y$ be image-open and image-nonmingled multifunction. Then F is l.e-c. iff F is u.e-c.*

Proof. ($\Rightarrow$) Let $x \in X$ and V be an open set with $F(x) \subseteq V$. Since F is image-open and l.e-c. Then $F^-(F(x))$ is E-open in X and $x \in F^-(F(x))$. Thus we have an e-open set U containing x such that $U \subseteq F^-(F(x))$ Therefore, $F(U) \subseteq F(F^-(F(x))) = F(x) \subseteq V$ by above note. This shows that F is u.e-c.

($\Leftarrow$) Let $x \in X$ and V be an open set with $F(x) \cap V \neq \varnothing$. Then $F^+(F(x))$ is E-open in X and $x \in F^+(F(x))$. Thus we have an e-open set U containing x such that $U \subseteq F^+(F(x))$. Therefore, if $z \in U$, then $F(z) \subseteq F(U) \subseteq F(x)$ and so $F(z) \cap V \neq \varnothing$. This shows that F is l.e-c. □

Now we will give some characterizations for e-continuity.

Theorem 3.8. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) F is l.e-c.;

(2) For every closed set $K \subseteq Y$ satisfying the condition $x \notin F^+(K)$, there exists an e-open set U containing x such that $U \cap F^+(K) = \emptyset$;

(3) For each closed subset K of Y, $eClF^{+}(K) \subseteq F^{+}(K)$;
(4) For any subset B of Y, $eClF^{+}(B) \subseteq F^{+}(ClB)$;
(5) For any subset B of Y, $F^{-}(IntB) \subseteq eIntF^{-}(B)$;
(6) For each open subset V of Y, $F^{-}(V)$ is E-open;
(7) For each closed subset K of Y, $F^{+}(K)$ is E-closed;
(8) For any subset A of X, $F(eClA) \subseteq ClF(A)$
(9) $F : (X, \tau_e) \to (Y, \sigma)$ is l.s.c.

Proof. (1)⇒(2): Let K be a closed set with $x \notin F^{+}(K)$. Then $Y - K$ is an open set and $F(x) \cap (Y - K) \neq \emptyset$. Since F is l.e-c., there exist an e-open set U containing x such that $F(z) \cap V \neq \emptyset$ for all $z \in U$ i.e. $U \subseteq F^{-}(Y - K)$. Therefore, we have $U \subseteq X - F^{+}(K)$ and so $U \cap F^{+}(K) = \emptyset$.

(2)⇒(3): Let K be a closed set with $x \notin F^{+}(K)$. By (2), there exist an e-open set U containing x such that $U \cap F^{+}(K) = \emptyset$ and hence $x \notin eClF^{+}(K)$. This shows that $eClF^{+}(K) \subseteq F^{+}(K)$.

(3)⇒(4): Let B be any subset of Y. Then ClB is a closed set and by (3), $eClF^{+}(ClB) \subseteq F^{+}(ClB)$. Therefore, we have $eClF^{+}(B) \subseteq F^{+}(ClB)$.

(4)⇔(5): These follow from the facts that $F^{-}(Y \backslash K) = X \backslash F^{+}(K)$, $Y \backslash ClB = Int(Y \backslash B)$ for $B \subseteq Y$ and $X \backslash eClA = eInt(X \backslash A)$ for $A \subseteq X$.

(5)⇒(6) Let V be an open set in Y. Then by (5), we have $F^{-}(V) = F^{-}(IntV) \subseteq eIntF^{-}(B) = \cup\{U : U$ is e-open and $U \subseteq F^{-}(B)\} \subseteq F^{-}(B)$. Hence we obtain that $F^{-}(B)$ is an E-open set in X.

(6)⇔(7) These follow from the fact that $F^{-}(Y \backslash K) = X \backslash F^{+}(K)$.

(7)⇒(8) Let A be any subset of X and K be any closed in Y containing $F(A)$. By (7), $F^{+}(K)$ is E-closed in X containing A. Hence $eClA \subseteq eClF^{+}(K) = Cl_{X_e}F^{+}(K) = F^{+}(K)$ and it follows that $F(eClA) \subseteq K$. Since this is true for each closed set K containing $F(A)$, we have $F(eClA) \subseteq ClF(A)$.

(8)⇒(7) Let $K \subseteq Y$ be a closed set. Since we always have $F(F^{+}(K)) \subseteq K$, $ClF(F^{+}(K)) \subseteq ClK$ and by (8), $F(eClF^{+}(K)) \subseteq ClF(F^{+}(K)) \subseteq ClK = K$. Therefore, $eClF^{+}(K) \subseteq F^{+}(K)$ and so $F^{+}(K)$ is E-closed in X.

(6)⇒(1) Let $x \in X$ and V be any open set with $F(x) \cap V = \emptyset$. Then by (6), $F^{-}(V)$ is an E-open set and it satisfies $x \in F^{-}(V)$. Therefore, we can find an e-open set U such that $x \in U \subseteq F^{-}(V)$. This completes the proof.

(1)⇔(9) It is obvious. □

Theorem 3.9. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) F is u.e-c.;
(2) For every closed set $K \subseteq Y$ satisfying the condition $F(x) \cap K = \emptyset$, there exists a e-open set U containing x such that $U \cap F^{-}(K) = \emptyset$;
(2) For each open subset V of Y, $F^{+}(V)$ is E-open;
(3) For each closed subset K of Y, $F^{-}(K)$ is E-closed;
(4) $F : (X, \tau_e) \to (Y, \sigma)$ is u.s.c.

The proof is similar to that of Theorem 3.8, and is omitted.

Definition 3.3. [2] The net $(x_\alpha)_{\alpha \in I}$ is e-convergent to x if for each e-open set U containing x, there exists an $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$.

Theorem 3.10. *The multifunction $F : X \to Y$ is l.e-c. (resp. u.e-c.) at $x \in X$ iff for each net $(x_\alpha)_{\alpha \in I}$ e-convergent to x and for each open subset V of Y with $F(x) \cap V \neq \emptyset$ (resp. $F(x) \subseteq V$), there is an $\alpha_0 \in I$ such that $F(x_\alpha) \cap V \neq \emptyset$ (resp. $F(x_\alpha) \subseteq V$) for all $\alpha \geq \alpha_0$.*

Proof. We prove only for lower e-continuity. The other is entirely analogous.

($\Rightarrow$): Let $(x_\alpha)_{\alpha \in I}$ be a net which e-converges to x in X and let V be any open set in Y such that $F(x) \cap V \neq \emptyset$. Since F is l.e-c., there exists an e-open set U in X containing x such that $U \subseteq F^-(V)$. Since (x_α) e-converges to x, it follows that there exists an $\alpha_0 \in I$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. Therefore, we obtain that $x_\alpha \in F^-(V)$ for all $\alpha \geq \alpha_0$. Thus, we have $F(x_\alpha) \cap V \neq \emptyset$ for all $\alpha \geq \alpha_0$.

($\Leftarrow$): Suppose that F is not l.e-c. Then there is an open set V in Y with $x \in F^-(V)$ such that for each e-open set U of X containing x, $x \in U \not\subseteq F^-(V)$ i.e. there is a $x_U \in U$ such that $x_U \notin F^-(V)$ for all U. Define $D = \{(x_U, U) : U \in eO(X), x_U \in U, x_U \notin F^-(V)\}$. Now the order " $\leq$ " defined by $(x_{U_1}, U_1) \leq (x_U, U) \Leftrightarrow U \subseteq U_1$ is a direction on D and g defined by $g : D \longrightarrow X$, $g((x_U, U)) = x_U$ is a net on X. The net $(x_U)_{(x_U,U) \in D}$ is e-convergent to x. But $F(x_U) \cap V = \emptyset$ for all $(x_U, U) \in D$. This is a contradiction. □

4. Some Properties of e-continuous Multifunctions

Let $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ be two multifunctions. In this case, for each $V \subseteq Z$, $(G \circ F)^+(V) = F^+(G^+(V))$ and $(G \circ F)^-(V) = F^-(G^-(V))$ where $G \circ F$ is the composition of F and G.

Theorem 4.1. *If the multifunctions $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are u(l).e-c., then the multifunction $G \circ F : X \rightrightarrows Z$ is u(l).e-c.*

Proof. We prove only for the case of upper e-continuity, the other is similar. Let W be any open set in Z. Since G is u.e-c., $G^+(W)$ is an E-open set in Y. Then $G^+(W)$ is an open set in Y and by upper e-continuity of F, we have $F^+(G^+(W)) = (G \circ F)^+(W)$ is an E-open set in X. This shows that $G \circ F$ is u.e-c. □

Theorem 4.2. *If the multifunction $F : X \rightrightarrows Y$ is u(l).e-c. and $G : Y \rightrightarrows Z$ is u(l).s.c., then the multifunction $G \circ F : X \rightrightarrows Z$ is u(l).e-c.*

Proof. This is similar to that of Theorem 4.1 and is omitted. □

Proposition 4.3. *(1)* [1] *If U is e-open in the space X and $A \subseteq X$ is open or dense, then $U \cap A$ is e-open in the subspace A.*

(2) [2] *Let S be a clopen subspace of a topological space X and $A \subseteq S$. Then A is e-open (e-closed) in S iff it is e-open (e-closed) in X.*

Theorem 4.4. *If $F : X \rightrightarrows Y$ is u(l).e-c. and $A \subseteq X$ is open or dense, then $F|_A : A \rightrightarrows Y$ is u(l).e-c.*

The proof is obvious from the above Proposition 4.3 (1) and we omit it.

Theorem 4.5. *Let $\{S_\alpha : \alpha \in I\}$ be clopen cover of X. Then a multifunction $F : X \rightrightarrows Y$ is u(l).e-c. iff the restrictions $F|_{S_\alpha} : S_\alpha \rightrightarrows Y$ are u(l).e-c. for every $\alpha \in I$.*

Proof. ($\Rightarrow$) Theorem 4.4.

($\Leftarrow$) We prove for the lower e-continuity. The other is analogous.

Let $x \in X$ and V be any open set with $F(x) \cap V \neq \emptyset$. Since $\{S_\alpha : \alpha \in \Lambda\}$ is a cover of X, there exists an $\alpha_0 \in \Lambda$ such that $x \in S_{\alpha_0}$. By hypothesis, there exists an e-open set U in S_{α_0} containing x such that $U \subseteq \left(F|_{S_{\alpha_0}}\right)^{-}(V)$. Then U is an e-open set in X containing x from Proposition 4.3 (2). Moreover, since $U \subseteq \left(F|_{S_{\alpha_0}}\right)^{-}(V) \subseteq F^{-}(V)$, we have that F is l.e-c. at x. $\square$

Theorem 4.6. *If the multifunctions $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Y$ are u(l).e-c., then the multifunction $F_1 \cup F_2 : X \rightrightarrows Y$, $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$ is u(l).e-c.*

Proof. We first prove for upper e-continuity. If $x \in X$ and V is an open set with $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x) \subseteq V$, then $F_1(x) \subseteq V$ and $F_2(x) \subseteq V$. By upper e-continuities of F_1 and F_2, there exist e-open sets U_1 and U_2 containing x such that $F_1(U_1) \subseteq V$ and $F_2(U_2) \subseteq V$. In this case, if $U := U_1 \cap U_2$ is selected, we have $(F_1 \cup F_2)(U) = F_1(U) \cup F_2(U) \subseteq V$. This shows that $F_1 \cup F_2$ is u.e-c.

Now let's we prove for lower e-continuity. Let $x \in X$ be an arbitrary point and V be any open set which satisfies $(F_1 \cup F_2)(x) \cap V \neq \emptyset$. In this case, there are three situations.

If $F_1(x) \cap V \neq \emptyset$, then there exists an e-open set U containing x such that $z \in U$ implies $F_1(z) \cap V \neq \emptyset$ and so $(F_1 \cup F_2)(z) \cap V \neq \emptyset$. This completes the proof.

If $F_2(x) \cap V \neq \emptyset$, then the proof is made similar to the previous case.

If $F_1(x) \cap V \neq \emptyset$ and $F_2(x) \cap V \neq \emptyset$, then a similar proof is made. $\square$

Theorem 4.7. *If Y is a normal space and the multifunctions $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$ are u.e-c., point closed and $F_1(x) \cap F_2(x) \neq \emptyset$ for all $x \in X$, then the multifunction $F_1 \cap F_2 : X \rightrightarrows Y$, $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$ is u.e-c.*

Proof. Let $x \in X$ and V be any open set with $(F_1 \cap F_2)(x) \subseteq V$. Then we have $F_1(x) \cap F_2(x) \cap (Y \backslash V) = \emptyset$. Since $F_1(x)$ and $F_2(x) \cap (Y \backslash V)$ are disjoint closed sets and Y is normal, there exist open sets V_1 and V_2 such that $F_1(x) \subseteq V_1$ and $F_2(x) \cap (Y \backslash V) \subseteq V_2$. If $V_3 := V_2 \cup V$, then we get $F_2(x) \subseteq V_3$. Since F_1 and F_2 are u.e-c., there exists e-open sets U_1 and U_2 containing x such that $F_1(U_1) \subseteq V_1$ and $F_2(U_2) \subseteq V_3$. Hence we have

$$(F_1 \cap F_2)(U) = F_1(U) \cap F_2(U) \subseteq V_1 \cap V_3 = V_1 \cap (V_2 \cup V) \subseteq V$$

for $U := U_1 \cap U_2$. This completes the proof. $\square$

We know that for the multifunctions $F_1 : X \rightrightarrows Y$ and $F_2 : X \rightrightarrows Z$, and for any subsets $B \subseteq Y$ and $C \subseteq Z$, the following equations are true:

(1) $(F_1 \times F_2)^{+}(B \times C) = F_1^{+}(B) \cap F_2^{+}(C)$,

(2) $(F_1 \times F_2)^{-}(B \times C) = F_1^{-}(B) \cap F_2^{-}(C)$.-

Theorem 4.8. *Let X, Y and Z be topological spaces and $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be multifunctions. If the multifunction $H : X \rightrightarrows Y \times Z$, $H(x) = F(x) \times G(x)$ is u(l).e-c., then F and G are u(l).e-c.*

Proof. Let $x \in X$ be an arbitrary point and V, W be open sets such that $F(x) \subseteq V$ and $G(x) \subseteq W$. Then we get $H(x) = F(x) \times G(x) \subseteq V \times W$. Since H is u.e-c., there exists an e-open set U containing x such that $U \subseteq H^{+}(V \times W)$. Therefore, we have $U \subseteq F^{+}(V) \cap G^{+}(W)$ and so $U \subseteq F^{+}(V)$ and $U \subseteq G^{+}(W)$. This completes the proof.

The proof in the case of lower e-continuity can be done similarly to the above. $\square$

For a multifunction $F : X \rightrightarrows Y$, the graph multifunction is defined as $G_F : X \rightrightarrows X \times Y$, $G_F(x) = \{x\} \times F(x)$.

Corollary 4.9. *A multifunction $F : X \rightrightarrows Y$ is u(l).e-c. if the graph multifunction G_F is u(l).e-c.*

Theorem 4.10. *Let X and X_α be topological spaces for $\alpha \in \Lambda$. If a multifunction $F : X \rightrightarrows \prod_{\alpha\in\Lambda} X_\alpha$ is u(l).e-c., then $p_\alpha \circ F$ is u(l).e-c., where $p_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \rightrightarrows X_\alpha$ is the projection for each $\alpha \in \Lambda$.*

Proof. We prove for upper e-continuity. The other can be done similarly.

Let $\alpha \in \Lambda$ be an arbitrary index and $x \in X$ be an arbitrary point. Suppose that $V_\alpha \subseteq Y_\alpha$ is an open set with $(p_\alpha \circ F)(x) \subseteq V_\alpha$. Then we get $x \in (p_\alpha \circ F)^{+}(V_\alpha) = F^{+}(p_\alpha^{-1}(V_\alpha)) = F^{+}(V_\alpha \times \prod_{\beta\neq\alpha} X_\beta)$. Since $V_\alpha \times \prod_{\beta\neq\alpha} X_\beta$ is open in $\prod_{\alpha\in\Lambda} X_\alpha$ and F is u.e-c., there exists an e-open set U containing x such that $U \subseteq F^{+}(V_\alpha \times \prod_{\beta\neq\alpha} X_\beta) = (p_\alpha \circ F)^{+}(V_\alpha)$. This shows that $p_\alpha \circ F$ is u.e-c. $\square$

Theorem 4.11. *If the multifunction $F : X \rightrightarrows Y$ is u.e-c. and Y is a normal space, then the multifunction $ClF : X \rightrightarrows Y$, $(ClF)(x) = ClF(x)$ is u.e-c.*

Proof. Let $x \in X$ and V be any open set with $ClF(x) \subseteq V$. Since Y is normal, there exits an open set G such that $ClF(x) \subseteq G \subseteq ClG \subseteq V$. Then we have $F(x) \subseteq G$. Since F is u.e-c., there exists an e-open set U containing x such that $F(U) \subseteq G$. Therefore, we have $ClF(x) \subseteq ClG \subseteq V$ and this shows that ClF is u.e-c. $\square$

Theorem 4.12. *A multifunction $F : X \rightrightarrows Y$ is l.e-c. iff $ClF : X \rightrightarrows Y$, $ClF(x) = ClF(x)$ is l.e-c.*

Proof. Let $x \in X$ and V be any open set with $ClF(x) \cap V \neq \emptyset$. Then $F(x) \cap V \neq \emptyset$ since V is open. By the hypothesis, there exists an e-open set U containing x such that $z \in U$ implies $F(z) \cap V \neq \emptyset$ and so $ClF(z) \cap V \neq \emptyset$. This shows that ClF is l.e-c.

Conversely, If $x \in X$ and V is an open set which satisfies $F(x) \cap V \neq \emptyset$, then $ClF(x) \cap V \neq \emptyset$. By the hypothesis, there exists an e-open set U containing x such that $z \in U$ implies $ClF(z) \cap V \neq \emptyset$ and so $F(z) \cap V \neq \emptyset$ because V is open. This shows that F is l.e-c. $\square$

5. Some Preservation properties

Definition 5.1. A space X is called e-Hausdorff [1] (resp. ultra Hausdorff [26]), if every two different points of X can be separated by two disjoint e-open (resp. clopen) sets.

It is noted in [1] that A topological space X is e-Hausdorff if and only if it is ultra Hausdorff.

Theorem 5.1. *Let Y be a normal topological space and $F : X \rightrightarrows Y$ be a multifunction which satisfies $F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$. If F is image-closed and u.e-c., then X is ultra Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a normal space, then there exists disjoint open sets V_1 and V_2 such that $F(x) \subseteq V_1$ and $F(y) \subseteq V_2$. By upper e-continuity of F, there exist e-open sets U_1 and U_2 containing x and y, respectively such that $U_1 \subseteq F^+(V_1)$ and $U_2 \subseteq F^+(V_2)$. Hence U_1 and U_2 are disjoint e-open sets, and so we have that X is e-Hausdorff i.e. ultra Hausdorff space. □

Theorem 5.2. *Let Y be a normal topological space. If $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ are image-closed and u.e-c. multifunctions, then the set $A = \{x : F(x) \cap G(x) \neq \emptyset\}$ is e-closed in X.*

Proof. Let $x \in X - A$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions and Y is a normal space, then there exist disjoint open sets V_1 and V_2 containing $F(x)$ and $G(x)$, respectively. Since F and G are u.e-c., there exist e-open sets U_1 and U_2 containing x such that $U_1 \subseteq F^+(V_1)$ and $U_2 \subseteq G^+(V_2)$. Put $U := U_1 \cap U_2$. Then U is an e-open set containing x and $U \cap A = \emptyset$ i.e. $U \subseteq X - A$. This shows that A is e-closed in X. □

Definition 5.2. A space X is called e-compact [1] (resp. e-Lindelöf [2]) if every e-open cover of X has a finite (resp. countable) subcover.

Theorem 5.3. *Let $F : X \to Y$ be an image-compact and u.e-c. multifunction. If X is e-compact (resp. e-Lindelöf) and F is surjective, then Y is compact (resp. Lindelöf).*

Proof. Let Φ be an open cover of Y. If $x \in X$, then we have $F(x) \subseteq \cup\Phi$. Thus Φ is an open cover of $F(x)$. Since $F(x)$ is compact, there exists a finite subfamily Φ_x of Φ such that $F(x) \subseteq \cup\Phi_x = V_x$. Then V_x is an open set in Y. Since F is u.e-c., there exists an e-open set U_x containing x such that $U_x \subseteq F^+(V_x)$. Therefore, $\Omega = \{U_x : x \in X\}$ is an e-open cover of X. Since X is e-compact, there exists points $x_1, x_2, ..., x_n \in X$ such that $X \subseteq \cup\{U_{x_i} : x_i \in X, i = 1, 2, ..., n\} \subseteq \cup\{F^+(V_{x_i}) : x_i \in X,\ i = 1, 2, ..., n\}$. So we obtain $Y = F(X) \subseteq F(\cup\{F^+(V_{x_i}) : i = 1, 2, ..., n\}) \subseteq \cup\{V_{x_i} : i = 1, 2, ..., n\} \subseteq \cup\{\Phi_{x_i} : i = 1, 2, ..., n\}$. Thus Y is compact.

The proof for Lindelöfness is similar and has been omitted. □

Definition 5.3. A topological space X is said to be e-separable if it contains a countable subset A with $eClA = X$.

Theorem 5.4. *Let $F : X \to Y$ be an image-separable and l.e-c. multifunction. If X is e-separable and F is surjective, then Y is separable.*

Proof. Let X be e-separable and A be countable subset of it such that $eClA = X$. Then by Theorem 3.8 (8), $Y = F(X) = F(eClA) \subseteq ClF(A)$. On the other hand, since $F(x)$ is separable for each $x \in A$, there is a countable subset A_x of it whose closure in the subspace $F(x)$ equal to $F(x)$. Then $\cup_{x\in A}A_x$ is a countable subset of Y and since $Cl(\cup_{x\in A}A_x) \supset \cup_{x\in A}ClA_x \supseteq \cup_{x\in A}F(x) = F(A)$, we have $Cl(\cup_{x\in A}A_x) \supseteq ClF(A) = Y$. This completes the proof. □

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