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OPTIMAL CONTROL OF A FRICTIONLESS UNILATERAL CONTACT PROBLEM FOR LOCKING MATERIALS

AREZKI TOUZALINE

Abstract. In this paper, we consider a frictionless contact with unilateral constraints associated to the normal compliance between a locking material and a deformable foundation. The goal is to study an optimal control problem which consists of leading the stress tensor as close as possible to a given target, by acting with a control on the boundary of the body. We study a continuous dependence on the data and we state an optimal control problem which admits at least one solution.

1. Introduction

Contact problems involving deformable bodies abund in industry and every daily life. The modelling, numerical analysis and computer simulations of such problems have been extensively studied in engineering and mathematical literature. See for instance [10, 13, 17, 22, 24]. The theory of variational inequalities is a powerful mathematical tool to represent various nonlinear value problems and mathematical models arising in Contact Mechanics. It is very developed. It is based on arguments of monotonicity and convexity, including properties of subdifferential of a convex function. References in the field are [2, 6, 7, 12, 15, 16, 25, 26]. In [25], the authors have studied the optimal control of quasivariational inequalities with applications of contact problems. In the same way, in [26] the authors have studied the hemivariational inequalities with applications of contact problems. In [28, 29], the authors have studied the asymptotic analysis of static elastic problems. The study of optimal control problems is very interesting for important applications in Physics, Mechanics, Automation and Systems Theory. For instance the theory of optimal control in the study of mathematical models of contact is quite limited. The difficulties are generated by the strong nonlinearities which arise in the boundary included in such models, also by some features like non-convexity and non-differentiability. Results on optimal control of various contact problems could be found in [1, 3, 4, 5, 11, 14, 18, 23, 26, 27] and the references therein. In mechanics, we say that a material is a locking material if it is deformed under the effect of an external force but the deformation stops once it reaches a certain value "M_L". After that, for any external force, the material cannot be deformed. The material is elastic if the deformation remains bounded. It returns back to its initial shape if we stop to exercise any external force on it. Locking materials belong to a class of hyperelastic materials which the strain

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tensor is constrained to stay in a given convex set. The study of elastic materials with locking effect was first introduced in [19, 20, 21]. There, the constitutive law of such materials was derived and different mechanical interpretations have been presented. Recall that the study of variational problems of locking materials was introduced in [8, 9]. In [8], the modeling of torsion of a cylindrical bar made of locking material was studied.

In this work we study an optimal control of a contact problem for elastic locking materials. The contact is frictionless with unilateral constraint associated to the normal compliance between a locking body and a deformable foundation.

In this paper, we claim to continue the research begun in [27]. We establish the variational formulation of the model. It is written in a form of an elliptic variational inequality in which the unknown is the displacement field and the data are the densities of applied forces φ_0, φ. Moreover, we study the continuous dependence of the solution with respect to the densities of applied forces. Also, we denote by Problem C1 the boundary optimal control problem which concerns this model. It consists of minimizing a cost functional which is convex and continuous. The control problem concerns the acting of a surfacic load in a part of the boundary in order to approach a given target σ_d by the stress tensor.

The paper is structured as follows. In Section 2 we introduce some notation, describe the mechanical problem and prove its weak solvability, Theorem 2.1. Section 3 is dedicated to a convergence result of the continuous dependence of the solution, Theorem 3.1. In section 4, we state the boundary optimal control C1 and prove that it has at least one solution, Theorem 4.1.

2. Setting of the problem

2.1. **The contact problem.** We consider a locking body which initially occupies a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently smooth boundary $\partial\Omega = \Gamma$ partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The body is clamped on Γ_1 and then the displacement vanishes there. It is acted upon by a volume force of density φ_0 in Ω and a surface traction of density φ on Γ_2. On Γ_3 the body is in frictionless contact with unilateral constraint and normal compliance with a deformable foundation.Then, the classical formulation of the mechanical problem written in terms of displacement is as follows.

Problem P_1. Find a displacement field $u : \Omega \to \mathbb{R}^d$ such that

$$div\sigma\,(u) = -\varphi_0 \text{ in } \Omega, \tag{2.1}$$

$$\sigma\,(u) \in F\varepsilon\,(u) + \partial I_B\,(\varepsilon\,(u)) \;\text{ in } \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on } \;\Gamma_1, \tag{2.3}$$

$$\sigma\,(u)\,\nu = \varphi \quad \text{ on } \Gamma_2, \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu \le g,\, \sigma_\nu + p\,(u_\nu) \le 0,\; (\sigma_\nu + p\,(u_\nu))\,(u_\nu - g) = 0 \\ \sigma_\tau\,(u) = 0 \end{array}\right\} \text{on } \Gamma_3. \tag{2.5}$$

Here, we denote by $\sigma\,(u)$ the stress field and $\varepsilon\,(u)$ the strain tensor. Equation (2.1) represents the equilibrium equation. Equation (2.2 represents the elastic constitutive law where F is a given nonlinear function and ∂I_B stands for the convex subdifferential of the indicator function of the set $B = \{\xi \in S_d;\ |\xi| \leq M_L\}$ which is defined as

$$\begin{cases} I_B(\xi) = 0, & \text{if } \xi \in B, \\ I_B(\xi) = +\infty, & \text{if } \xi \notin B \end{cases} \quad \text{for } \xi \in S_d.$$

(2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma(u)\nu$ represents the normal stress vector. The Condition (2.5) is the classical Signorini contact condition associated to the normal compliance p; it represents a combination of the Signorini contact for contact with a rigid foundation and the normal compliance condition for contact with a deformable foundation. It models the contact with a foundation made of a rigid body covered with a soft layer of deformable material of thickeness g. The inequality $u_\nu \leq g$ restricts the allowed penetration. Details and various mechanical interpretations can be found in [17]. The nullity of the tangential condition (2.5) represents the frictionless contact. Recall that the inner products "." and the corresponding norms on $\mathbb{R}^d$ and S_d are given by

$$u.v = u_i v_i, \quad |v| = (v.v)^{\frac{1}{2}} \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \ |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in S_d,$$

where S_d is the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted.

2.2. **Variational Formulation.** Now, to proceed with the variational formulation, we need the following function spaces:

$$H = \left(L^2(\Omega)\right)^d, \ H_1 = \left(H^1(\Omega)\right)^d, \ Q = \left\{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products:

$$(u, v)_H = \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \text{ where } \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ and } u_{i,j} = \frac{\partial u_i}{\partial x_j};$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every element $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Also, for a regular function (say C^1) $\sigma \in Q$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

and we recall that the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.v da \quad \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\}.$$

We define the closed convex set of admissible displacements by

$$K_1 = \{v \in V;\, v_\nu \leq g \ a.e. \text{ on } \Gamma_3\},$$

where $g \geq 0$ represents the maximum value of the penetration and the closed convex subset of V by

$$K_2 = \{v \in V;\, \varepsilon(v(x)) \in B \ a.e.\ x \in \Omega\}.$$

Next, since $meas(\Gamma_1) > 0$, the following Korn's inequality holds [10],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.6}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (2.6) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ such that

$$\|v\|_{(L^2\Gamma)^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{2.7}$$

We assume that the body forces and surface tractions have the regularity

$$\varphi_0 \in H, \qquad \varphi \in \left(L^2(\Gamma_2)\right)^d. \tag{2.8}$$

Next, in the study of Problem P_1 we assume that the elasticity operator F satisfies

$$\left.\begin{array}{l}
(a)\ F : \Omega \times S_d \to S_d;\\
(b) \text{ there exists } M > 0 \text{ such that}\\
|F(x, \varepsilon_1) - F(x, \varepsilon_2)| \leq M |\varepsilon_1 - \varepsilon_2|,\ \forall \varepsilon_1, \varepsilon_2 \in S_d,\\
\qquad a.e.\ x \in \Omega;\\
(c) \text{ there exists } m > 0 \text{ such that}\\
\qquad (F(x, \varepsilon_1) - F(x, \varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m |\varepsilon_1 - \varepsilon_2|^2,\\
\qquad \forall \varepsilon_1, \varepsilon_2 \in S_d,\ a.e.\ x \in \Omega;\\
(d) \text{ the mapping } x \to F(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega,\\
\qquad \text{for any } \varepsilon \in S_d;\\
(e)\ F(x, 0) = 0 \text{ for } a.e.\ x \in \Omega.
\end{array}\right\} \tag{2.9}$$

The normal compliance function satisfies

$$\begin{array}{l}
(a)\ \ p : \mathbb{R} \to \mathbb{R}_+;\\
(b)\ \ \exists L_p > 0 \text{ such that}\\
|p(r_1) - p(r_2)| \leq L_p |r_1 - r_2|,\ \forall r_1,\, r_2 \in \mathbb{R};\\
(c)\ \ (p(r_1) - p(r_2))(r_1 - r_2) \geq 0,\ \forall r_1,\, r_2 \in \mathbb{R};\\
(d)\ \ p(r) = 0, \text{ for all } r \leq 0.
\end{array} \tag{2.10}$$

We derive now the variational formulation of Problem P_1. To this end, let $(u, \sigma(u))$ be a pair of smooth functions which satisfies (2.1) - (2.5). Let $v \in V$.

Multiplying the equilibrium equation (2.1) by $v-u$ and use the Green formula, we deduce that

$$(\sigma(u),\varepsilon(v)-\varepsilon(u))_Q=(\varphi_0,v-u)_H+\int_\Gamma \sigma(u)\nu.(v-u)\,da.$$

Using the boundary conditions (2.3), (2.4) and (2.5), we have

$$\begin{aligned}&(\sigma(u),\varepsilon(v)-\varepsilon(u))_Q\\&=(\varphi_0,v-u)_H+(\varphi,v-u)_{(L^2(\Gamma_2))^d}+\textstyle\int_{\Gamma_3}\sigma_\nu(u)(v_\nu-u_\nu)\,da.\end{aligned}\tag{2.11}$$

By the condition (2.5), we have

$$\begin{aligned}\textstyle\int_{\Gamma_3}\sigma_\nu(u)(v_\nu-u_\nu)\,da&=\textstyle\int_{\Gamma_3}(\sigma_\nu(u)+p(u_\nu))(v_\nu-u_\nu)\,da\\&\quad-\textstyle\int_{\Gamma_3}p(u_\nu))(v_\nu-u_\nu)\,da\end{aligned}$$

As for $v\in K_1$, we have

$$\int_{\Gamma_3}(\sigma_\nu(u)+p(u_\nu))(v_\nu-u_\nu)\,da\geq 0,$$

then, we deduce that

$$\int_{\Gamma_3}\sigma_\nu(u)(v_\nu-u_\nu)\,da\geq-\int_{\Gamma_3}p(u_\nu))(v_\nu-u_\nu)\,da.\tag{2.12}$$

From the constitutive law (2.2), we have

$$\sigma(u)=F\varepsilon(u)+\varsigma(u)\ \text{ and }\varsigma(u)\in\partial I_B(\varepsilon(u))\ \text{ in }\Omega.$$

This relation, for $v,u\in K_2$, implies

$$\varsigma(u).(\varepsilon(v)-\varepsilon(u))\leq I_B(\varepsilon(v))-I_B(\varepsilon(u))=0\text{ in }\Omega.$$

Hence, we obtain

$$(\sigma(u),\varepsilon(v)-\varepsilon(u))_Q\leq(F\varepsilon(u),\varepsilon(v)-\varepsilon(u))_Q.\tag{2.13}$$

Next, we denote $K=K_1\cap K_2$ and we define the operator $A:V\to V$ by

$$(Au,v)_V=(F\varepsilon(u),\varepsilon(v))_Q,\ \forall u,v\in V.\tag{2.14}$$

It follows from (2.9) that the operator A is strongly monotone and Lipschitz continuous as

$$\begin{aligned}&(a)\ \ (Au-Av,u-v)_V\geq m\|u-v\|_V^2,\ \forall u,v\in V,\\&(b)\ \ \|Au-Av\|_V\leq M\|u-v\|_V,\ \forall u,v\in V.\end{aligned}\tag{2.15}$$

Also, we define the functional $j:V\times V\to\mathbb{R}$ by

$$j(u,v)=\int_{\Gamma_3}p(u_\nu)v_\nu da,\ \forall u,v\in V.$$

Now, using Riesz's representation theorem, there exists $f\in V$ such that for all $v\in V$, we have

$$(f,v)_V=(\varphi_0,v)_H+(\varphi,v)_{(L^2(\Gamma_2))^d}.\tag{2.16}$$

Finally, inserting (2.14) and (2.12) in (2.11) and taking into account of (2.13) and (2.16), we obtain the following variational formulation of Problem P_1.

Problem P_2. Find $u\in K$ such that

$$(Au,v-u)_V+j(u,v-u)\geq(f,v-u)_V\ \ \forall v\in K.\tag{2.17}$$

Theorem 2.1. *Let (2.8), (2.9) and (2.10) hold. Then, there exists a unique solution of Problem P_2.*

Proof. We define the operator $(Cu,v)_V = (Au,v)_V + j(u,v)$, $\forall u,v \in V$; by (2.10) and (2.15), C is Lipschitz continuous and strongly monotone; then moreover by using (2.8), since K is a non-empty closed convex, it follows (see [23]) that the inequality (2.17) has a unique solution. □

3. A continuous dependence result

In this section, we study the dependence of the solution u of Problem P_2 with respect the data φ_0 and φ. To this end, we assume that the hypotheses (2.8) - (2.10) hold, and for each $n \in \mathbb{N}^*$, we consider a perturbation φ_0^n and φ^n of φ_0 and φ, respectively, which satisfy (2.8) - (2.10). Then, we consider the variational problem below.

Problem P_2^n. Find $u^n \in K$ such that

$$\begin{aligned}&(Au^n, v-u)_V + j(u^n, v-u^n)\\ &\geq (\varphi_0^n, v-u^n)_H + (\varphi^n, v-u^n)_{(L^2(\Gamma_2))^d} \quad \forall v \in K.\end{aligned} \tag{3.1}$$

It follows from Theorem 2.1 that, for each $n \in \mathbb{N}^*$, Problem P_2^n has a unique solution $u^n \in K$. Next, we study the behaviour of the solution u^n as $n \to +\infty$. Then, we have the following result.

Theorem 3.1. *Let that (2.8) - (2.10) hold and, moreover, assume*

$$\varphi_0^n \to \varphi_0 \text{ weakly in } H, \tag{3.2}$$

$$\varphi^n \to \varphi \text{ weakly in } \left(L^2(\Gamma_2)\right)^d. \tag{3.3}$$

Then, the following convergence holds

$$u^n \to u \text{ strongly in } V. \tag{3.4}$$

The proof of Theorem 3.1 will be carried out in several steps. In the first step, we must provide the following convergence result.

Lemma 3.2. *The following convergence holds*

$$u^n \to u \text{ weakly in } V. \tag{3.5}$$

Proof. We take $v = 0_V$ in (3.1) to obtain

$$(Au^n - A0_V, u^n)_V + j(u^n, u^n) \leq (\varphi_0^n, u^n)_H + (\varphi^n, u^n)_{(L^2(\Gamma_2))^d} - (A0_V, u^n)_V.$$

Then, using (2.7), (2.8), (2.10) and (2.15), we deduce that there exists a constant $c > 0$ such that

$$\|u^n\|_V \leq c\left(\|\varphi_0^n\|_H + \|\varphi^n\|_{(L^2(\Gamma_2))^d} + \|A0_V\|_V\right).$$

The convergences (3.2) and (3.3) imply that the sequences (φ_0^n) and (φ^n) are bounded in H and $\left(L^2(\Gamma_2)\right)^d$, respectively. Hence, we deduce that there exists a constant $c_1 > 0$ such that

$$\|u^n\|_V \leq c_1. \tag{3.6}$$

Then, using (3.6) and a standard compactness argument, we see that there exits $\tilde{u} \in V$ such that, passing to a subsequence, still denoted (u^n), we have

$$u^n \to \tilde{u} \text{ weakly in } V. \tag{3.7}$$

Moreover, K is a non-empty closed convex, then it is weakly closed convex, so $\tilde{u} \in K$.

Now, we etablish the following equality

$$\tilde{u} = u. \tag{3.8}$$

Indeed, we take $v = \tilde{u}$ in (3.1) to obtain that

$$(Au^n, u^n - \tilde{u})_V \leq (\varphi_0^n, u^n - \tilde{u})_H + (\varphi^n, u^n - \tilde{u})_{(L^2(\Gamma_2))^d} + j\left(u^n, \tilde{u}\right) - j\left(u^n, u^n\right).$$

Next, using (3.2), (3.3), (3.7) and the compactness of the operator trace, we pass to the upper limit in this inequality to obtain

$$\limsup_{n\to+\infty} (Au^n, u^n - \tilde{u})_V \leq 0.$$

On the other hand, using (3.7) and a standard pseudomonotonicity argument yield

$$\liminf_{n\to+\infty} (Au^n, u^n - \tilde{u})_V \geq (A\tilde{u}, \tilde{u} - v)_V \quad \forall v \in V. \tag{3.9}$$

Moreover, passing to the upper limit in (3.1) and using again (3.2), (3.3), (3.7) and the compactness of the trace operator, we get

$$\limsup_{n\to+\infty} (Au^n, u^n - v)_V \leq (\varphi_0, \tilde{u} - v)_H + (\varphi, \tilde{u} - v)_{(L^2(\Gamma_2))^d} + j\left(\tilde{u}, v - \tilde{u}\right) \quad \forall v \in K.$$

Then, we combine this inequality above and (3.9) to deduce that

$$\begin{aligned} &(A\tilde{u}, v - \tilde{u})_V + j\left(\tilde{u}, v - \tilde{u}\right) \\ &\quad \geq (\varphi_0, v - \tilde{u})_H + (\varphi, v - \tilde{u})_{(L^2(\Gamma_2))^d} \;\; \forall v \in K. \end{aligned} \tag{3.10}$$

Now, we take $v = \tilde{u}$ in (2.17) and $v = u$ in (3.10), adding the resulting inequalities and using assumptions (2.10)(c) and (2.15(a), we obtain that the equality (3.8) holds. □

Finally, we proceed with the following strong convergence result.

Lemma 3.3. *We have the following convergence*

$$u^n \to u \textit{ strongly in } V. \tag{3.11}$$

Proof. Let $n \in \mathbb{N}^*$. We take $v = u$ in (3.1) to get the following inequality

$$(Au^n, u^n - u)_V \leq (\varphi_0^n, u^n - u)_H + (\varphi^n, u^n - u)_{(L^2(\Gamma_2))^d} + j\left(u^n, u - u^n\right).$$

Next, with this inequality and (2.15)(a), we see that

$$\begin{aligned} &m \left\|u^n - u\right\|_V^2 \leq (Au^n - Au, u^n - u)_V \\ &= (Au^n - Au, u^n)_V - (Au^n - Au, u)_V \\ &= (\varphi_0^n, u^n - u)_H + (\varphi^n, u^n - u)_{(L^2(\Gamma_2))^d} + j\left(u^n, u - u^n\right) - (Au, u^n - u)_V. \end{aligned}$$

We now use (2.10), the boundedness of (u^n), (3.2), (3.3), (3.7) and the compactness of the trace operator to see that

$$\Big((\varphi_0^n, u^n - u)_H + (\varphi^n, u^n - u)_{(L^2(\Gamma_2))^d} + j\left(u^n, u - u^n\right) - (Au, u^n - u)_V\Big) \to 0 \text{ as } n \to +\infty.$$

Then, we obtain

$$\|u^n - u\|_V \to 0 \text{ as } n \to +\infty,$$

which concludes the proof. □

On the other hand, from mathematical point of view, the convergence result (3.4) means that the weak solution of Problem P_2 depends continuously on the densities of applied forces.

4. The boundary optimal control problem

We now suppose that $\varphi_0 \in H$ is fixed and consider the following state variational problem.

Problem $\mathbf{Q_1}$. For $\varphi \in \left(L^2\left(\Gamma_2\right)\right)^d$ (called control), find $u \in K$ such that

$$\begin{aligned} &(Au, v-u)_V + j\left(u, v-u\right) \\ &\geq \left(\varphi_0, v-u\right)_H + \left(\varphi, v-u\right)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \end{aligned} \tag{4.1}$$

Following the existence and uniqueness of Problem P_2, we deduce that for every control $\varphi \in \left(L^2\left(\Gamma_2\right)\right)^d$, the state variational problem Q_1 has a unique solution $u \in K$.

We define the cost functional

$$\mathcal{L} : V \times \left(L^2\left(\Gamma_2\right)\right)^d \to \mathbb{R}_+,$$

by

$$\mathcal{L}\left(u, \varphi\right) = \alpha \left\|u - u_d\right\|_V + \beta \left\|\varphi\right\|_{(L^2(\Gamma_2))^d}, \tag{4.2}$$

where $\alpha, \beta > 0$ and $u_d \in K$. We have that $\sigma_d = \sigma\left(u_d\right) = F\varepsilon\left(u_d\right)$, then for $u \in K$, we have $\sigma\left(u\right) = F\varepsilon\left(u\right)$, and $\left\|\sigma\left(u\right) - \sigma\left(u_d\right)\right\|_Q \leq M \left\|u - u_d\right\|_V$; so $\sigma\left(u\right)$ is a close of $\sigma\left(u_d\right)$.

Next, we denote the set of admissible pairs U_{ad} by

$$U_{ad} = \left\{ (u, \varphi) \in K \times \left(L^2\left(\Gamma_2\right)\right)^d, \text{ such that (3.1) is satisfied} \right\},$$

and we consider the following boundary optimal control problem.

Problem C1. Find $(u^*, \varphi^*) \in U_{ad}$ such that

$$\mathcal{L}\left(u^*, \varphi^*\right) = \min_{(u,\varphi)\in U_{ad}} \mathcal{L}\left(u, \varphi\right).$$

Theorem 4.1. *Assume that (2.8), (2.9) and (2.10)(c) hold. Then Problem C1 has at least one solution.*

Proof. Take $v = 0_V$ in (3.1), using (2.7) and (2.10)(c), we deduce that the solution u of Problem Q_1 is bounded in V as

$$\|u\|_V \leq \frac{c_0 \left(\left\|\varphi_0\right\|_H + d_\Omega \left\|\varphi\right\|_{(L^2(\Gamma_2))^d} \right)}{m},$$

where $c_0 > 0$. This inequality implies that

$$0 \leq \inf_{(u,\varphi)\in U_{ad}} \mathcal{L}\left(u, \varphi\right) < +\infty.$$

Then, there exists a sequence $(u^n, \varphi^n) \subset U_{ad}$ such that

$$\mathcal{L}\left(u^n, \varphi^n\right) \to \inf_{(u,\varphi)\in U_{ad}} \mathcal{L}\left(u, \varphi\right) \text{ as } n \to +\infty.$$

The sequence (u^n, φ^n) is bounded in $V \times \left(L^2(\Gamma_2)\right)^d$, so there exists an element

$$(u^*, \varphi^*) \in V \times \left(L^2(\Gamma_2)\right)^d$$

such that passing to a subsequence still denoted by (u^n, φ^n), it follows that as $n \to +\infty$,

$$\begin{cases} (a)\ \ u^n \to u^* \text{ weakly in } V, \\ (b)\ \ \varphi^n \to \varphi^* \text{ weakly in } \left(L^2(\Gamma_2)\right)^d. \end{cases} \tag{4.3}$$

Next for the rest of the proof, we have needed to prove that

$$u^n \to u^* \text{ strongly in } V \text{ as } n \to +\infty. \tag{4.4}$$

Indeed, as $(u^n, \varphi^n) \in U_{ad}$, u^n satisfies the inequality:

$$\begin{aligned} &(Au^n, v - u^n)_V + j(u^n, v - u^n) \\ &\geq (\varphi_0, v - u^n)_H + (\varphi^n, v - u^n)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \end{aligned} \tag{4.5}$$

Using (2.10)(c) and (4.5), we deduce that

$$\begin{cases} m \left\| u^n - u^* \right\|_V^2 \leq (Au^n - Au^*, u^n - u^*)_V \\ \leq (Au^n, u^n - u^*)_V - (Au^*, u^n - u^*)_V \\ \leq (Au^*, u^* - u^n)_V + j(u^n, u^* - u^n) \\ + (\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d}. \end{cases} \tag{4.6}$$

Now from (4.3)(a) and (2.14), we have that $(Au^*, u^n - u^*)_V \to 0$ as $n \to +\infty$. Next, using that $u^n \to u^*$ weakly in V implies that $u^n \to u^*$ strongly in $\left(L^2(\Gamma_2)\right)^d$ and as (φ^n) is bounded in $\left(L^2(\Gamma_2)\right)^d$, then

$$j(u^n, u^* - u^n) + (\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d} \to 0 \text{ as } n \to +\infty.$$

Thus we deduce that the last member of the right hand side of the last inequality tends to zero. Hence from (4.6), we get (4.4). On the other hand, as K is a non-empty closed convex, then it is weakly closed convex and so $u^* \in K$. Moreover, using (4.3)(b), (4.4) and passing to the limit as $n \to +\infty$ in (4.5), we obtain that $(u^*, \varphi^*) \in U_{ad}$ and it is a solution of Problem C1. □

Conclusion. *In this paper we have studied the optimal control of a frictionless contact problem for locking materials with unilateral constraints associated to a normal compliance. We have obtained a convergence result which establishes the continuous dependence of the solution with respect to the densities of applied forces and proved that the boundary optimal control problem admits at least one solution.*

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Laboratory of Dynamical Systems, Faculty of Mathématics, USTHB, BP 32 EL Alia, Bab-Ezzouar, 16111, Algiers, Algeria

Email address: ttouzaline@yahoo.fr