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COMMON FIXED POINTS FOR GENERALIZED MULTIVALUED MAPPINGS VIA SIMULATION FUNCTION IN S-METRIC SPACES

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Abstract. The goal of this work is to introduce the notion of generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of mappings and establish common fixed point theorems for such mappings in complete S-metric spaces. Additionally, we deduce some corollaries from our main result and provide examples in support of the established results. With the help of the obtained result, sufficient conditions for the existence of common solution to the system of nonlinear integral equations are studied. The results obtained in this work extend and generalize several well-known results in the existing literature.

1. Introduction

Fixed point theory is one of the most useful branches of nonlinear analysis, providing fruitful tools for the resolution of many problems arising from different fields of pure mathematics, applied sciences, engineering and more. A fundamental result in fixed point theory is the Banach contraction principle [5]. This result has been extended in many directions for single and multivalued cases on a metric space. In 1969, *Nadler* [20] introduced the notion of multivalued contraction mappings and showed that such mappings have a fixed point in a complete metric space. After that, many fixed point theorems have been proved by various authors as a generalization of the *Nadler's* theorem (see [2, 4, 6, 8, 9, 12, 15, 17, 18]).

In 2017, *Gairola and Khantwal* [10] introduced the multi-valued contractions in S-metric spaces and proved some fixed point theorems for multi-valued maps on S-metric space. Their results extend and generalise the results of *Nadler* [20], *Sedghi et al.* [34] and others.

In 2015, *Khojasteh et al.* [14] introduced the notion of a simulation function with the aim of defining a new class of contractions called $\mathcal{Z}$-contractions. They studied the existence and uniqueness of fixed point for $\mathcal{Z}$-contractive type operators (see [14], Theorem 2.8). Using the idea of simulation functions, different contractive conditions can be expressed in a simple and unified way. A large number of authors

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have published many research papers using simulation functions (see, [3, 7, 13, 16, 21, 22, 23, 28, 29, 30, 31, 33, 36] and many others).

Recently, *Antal and Gairola* [1], introduced the notion of generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of mappings and establish common fixed point theorems for such mappings in the framework of complete metric spaces using simulation functions. In 2022, *Pourgholam et al.* [27] proved some common fixed point theorems for single valued and multi-valued mappings in S-metric spaces (see, also [26]).

Very recently, *Mlaiki et al.* [19] have studied simulation functions with S-metric spaces, which are recent generalizations of metric spaces, and established several fixed point theorems in these spaces, including an application to the fixed-circle problem.

In this paper, we introduce the notion of generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for a pair of mappings and establish some common fixed point theorems for such mappings in the setting of S-metric spaces using simulation function. We also provide an application for the validation of the established result.

2. Preliminaries

Let (Ω, d) be a metric space. In the sequel, we use the following notations:

$$CB(\Omega) := \{A : A\, is\, a\, nonempty\, closed\, and\, bounded\, subset\, of\, \Omega\},$$

$$D(x, A) := \inf\{d(x, y) : y \in A\},$$

$$H(A, B) := \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}.$$

The function H is a metric on $CB(\Omega)$ and is called the Hausdorff metric induced by the metric d. Also,

$$\delta(A, B) := \sup\{d(x, y) : x \in A,\, y \in B\}$$

and

$$D(A, B) := \inf\{d(x, y) : x \in A,\, y \in B\}.$$

Note: $\delta(A, A) = diam\,(A)$. If $diam\,(A) < \infty$, then A is bounded.

Definition 2.1. A point $x_0 \in \Omega$ is said to be a fixed point of the multi-valued mapping $\mathcal{T}$ if $x_0 \in \mathcal{T}x_0$.

Theorem 2.1. *([20]) Let (Ω, d) be a complete metric space and $\mathcal{T} : \Omega \to CB(\Omega)$ be a contraction mapping such that*

$$H(\mathcal{T}x, \mathcal{T}y) \le h\, d(x, y), \tag{2.1}$$

for all $x, y \in \Omega$ and for some $h \in [0, 1)$. Then there exists $x^* \in \Omega$ such that $x^* \in \mathcal{T}x^*$.

The notion of simulation function was introduced by *Khojasteh et al.* [14] as follows:

Definition 2.2. ([14]) Let $\zeta\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ be a mapping. Then ζ is called a simulation function if it fulfills the hypotheses given below:

(ζ_1) $\zeta(0,0)=0$;

(ζ_2) $\zeta(t,s)<s-t$ for all $s,t>0$;

(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $[0,\infty)$ such that $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0$, then $\limsup_{n\to\infty}\zeta(t_n,s_n)<0$.

Example 2.2. ([14]) Let $\phi_r\colon[0,\infty)\to[0,\infty)$ be continuous functions with $\phi_r(t)=0$ if and only if $t=0$. For $n=1,2,3,4,5$, we define the mappings $\zeta_n\colon[0,\infty)^2\to\mathbb{R}$, as follows:

(1) $\zeta_1(t,s)=\phi_1(s)-\phi_2(t)$ for all $t,s\in[0,\infty)$, where $\phi_1(t)<t\le\phi_2(t)$ for all $t>0$.

(2) $\zeta_2(t,s)=s-\frac{g_1(t,s)}{g_2(t,s)}$ for all $t,s\in[0,\infty)$, where $g_1,g_2\colon[0,\infty)^2\to[0,\infty)$ are two continuous functions with respect to each variable such that $g_1(t,s)>g_2(t,s)$ for all $t,s>0$.

(3) $\zeta_3(t,s)=s-\varphi(s)-t$ for all $t,s\in[0,\infty)$, where $\varphi\colon[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0$ if and only if $t=0$.

(4) Let $\varphi\colon[0,\infty)\to[0,1)$ be a function such that $\limsup_{t\to r^+}\varphi(t)<1$ for all $r>0$. We define $\zeta_4(t,s)=s\,\varphi(s)-t$ for all $t,s\in[0,\infty)$.

(5) Let $\eta\colon[0,\infty)\to[0,\infty)$ be an upper semi-continuous function such that $\eta(t)<t$ for all $t>0$ and $\eta(0)=0$. We define $\zeta_5(t,s)=\eta(s)-t$ for all $t,s\in[0,\infty)$.

Then ζ_n for $n=1,2,3,4,5$ are simulation functions.

Argoubi et al. [3] slightly modified the definition of simulation function by removing the condition (ζ_1).

Definition 2.3. ([3]) A simulation function is a mapping $\zeta\colon[0,\infty)\times[0,\infty)\to\mathbb{R}$ fulfilling the following axioms:

(ζ_2) $\zeta(t,s)=s-t$ for all $s,t>0$;

(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $[0,\infty)$ such that $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0$ and $t_n<s_n$, then $\limsup_{n\to\infty}\zeta(t_n,s_n)<0$.

The set of all simulation functions is denoted by $\mathcal{Z}$ [14]. In [14], the notion of a $\mathcal{Z}$-contraction was defined to generalize the Banach contraction principle as follows:

Definition 2.4. ([14]) Let (Ω,d) be a metric space and $\mathcal{T}\colon\Omega\to\Omega$ a mapping and $\zeta\in\mathcal{Z}$. Then $\mathcal{T}$ is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied for all $x,y\in\Omega$:

$$\zeta(d(\mathcal{T}x,\mathcal{T}y),d(x,y))\ge 0. \tag{2.2}$$

Every $\mathcal{Z}$-contraction mapping is contractive and hence continuous (see [7, 14, 29] for basic properties and some examples of a $\mathcal{Z}$-contraction). In [14], *Khojasteh et al.* used the notion of a simulation function to unify several existing fixed point results in the literature.

On the other hand, *Sedghi et al.* [34] introduced the notion of S-metric space as follows.

Definition 2.5. ([34]) Let Ω be a non-void set and $S\colon\Omega^3\to[0,+\infty)$ be a function fulfilling the hypotheses below for all $x,y,z,u\in\Omega$:

(a) $S(x,y,z) = 0$ if and only if $x = y = z$;
(b) $S(x,y,z) \leq S(x,x,u) + S(y,y,u) + S(z,z,u)$.
Then, (Ω, S) is called an S-MS and S is called an S-metric on Ω.

Example 2.3. ([27]) Let $\Omega = [0,+\infty)$ and $\alpha \geq 0$. Define $S\colon \Omega^3 \to [0,+\infty)$ by

$$S(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y,z\} - \alpha, & \textit{otherwise}. \end{cases}$$

Then S is an S-MS on Ω.

Example 2.4. ([34]) Let $\Omega = \mathbb{R}$ and $S(x,y,z) = |x-z| + |y-z|$. Then S is an S-metric on Ω, named the usual S-metric on Ω.

Lemma 2.5. *(*[34]*, Lemma 2.5) In an S-metric space (Ω, S), we have $S(x,x,y) = S(y,y,x)$ for all $x, y \in \Omega$.*

Definition 2.6. ([34]) Let (Ω, S) be an S-metric space. For $r > 0$ and $x \in \Omega$, the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r are defined as follows:

$$B_S(x,r) := \{y \in \Omega : \ S(y,y,x) < r\},$$
$$B_S[x,r] = \{y \in \Omega : \ S(y,y,x) \leq r\}.$$

Let τ be the collection of subsets $A \subset \Omega$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x,r) \subset A$. Then, τ is a topology on Ω, and A is said to be S-open. The complement of an S-open set is said to be S-closed.

Definition 2.7. ([34]) Let (Ω, S) be an S-metric space.
(1) If for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$, then the subset A is called an open subset of Ω.
(2) A subset A of Ω is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x, y \in A$.
(3) A sequence $\{x_n\}$ in Ω converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote this by $\lim_{n\to\infty} x_n = x$.
(4) A sequence $\{x_n\}$ in Ω is called Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$.
(5) The S-metric space (Ω, S) is said to be complete if every Cauchy sequence is convergent.

Example 2.6. ([34]) Let (Ω, S) be as in Example 2.4. Then (Ω, S) is complete.

Lemma 2.7. *(*[34]*, Lemma 2.12) Let (Ω, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, x_n, y_n) \to S(x,x,y)$ as $n \to \infty$.*

On the other hand, the relationship between a metric and an S-metric are given as follows:

Lemma 2.8. *(*[11]*) Let (Ω, d) be a metric space. Then, the following properties are satisfied:*
(1) $S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x, y, z \in \Omega$ is an S-metric on Ω.
(2) $x_n \to x$ in (Ω, d) if and only if $x_n \to x$ in (Ω, S_d).

(3) $\{x_n\}$ *is Cauchy in* (Ω, d) *if and only if* $\{x_n\}$ *is Cauchy in* (Ω, S_d).
(4) (Ω, d) *is complete if and only if* (Ω, S_d) *is complete.*

The metric S_d was called the S-metric generated by d (see [24]). Some examples of an S-metric that is generated by any metric are known (see [11, 24] for more details).

Lemma 2.9. *([32]) Let (Ω, S) be a complete S-metric space and $\{x_n\}$ is a sequence in Ω such that $\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that for every $k > 0$, corresponds to m_k, we can take n_k which is smallest such that $S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon$, $S(x_{m_k}, x_{m_k}, x_{n_k-1}) < \varepsilon$ and*

$$(i) \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon,$$

$$(ii) \lim_{k\to\infty} S(x_{m_k+1}, x_{m_k+1}, x_{n_k}) = \varepsilon,$$

$$(iii) \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k+1}) = \varepsilon.$$

Definition 2.8. ([27]) Let (Ω, S) be an S-metric space. Define $S_H\colon (CB(\Omega))^3 \to [0, +\infty)$ by

$$S_H(A, B, C) = H_S(A, C) + H_S(B, C),$$

where

$$H_S(A, B) = \max\{h_S(A, B), h_S(B, A)\},$$

$$h_S(A, B) = \sup\{S(a, a, B) : a \in A\}, \text{ and}$$

$$S(a, a, B) = \inf\{S(a, a, b) : b \in B\}.$$

For more details see [26].

Theorem 2.10. *([26]) S_H is an S-metric on $CB(\Omega)$.*

We call S_H the Hausdorff S-metric on $CB(\Omega)$ generated by S.

Lemma 2.11. *([10], Lemma 3.1) Let (Ω, S) be an S-metric space and $A, B \in CB(\Omega)$. Then for each $a \in A$, we have*

$$S(a, a, B) \leq S_H(A, A, B).$$

In [10], *Gairola and Khantwal* introduced the following.

Definition 2.9. ([10]) Let (Ω, S) be an S-metric space. A function $\mathcal{T}\colon \Omega \to CB(\Omega)$ is said to be a multi-valued contraction on Ω if there exists a constant $\alpha \in [0, 1)$ such that

$$S_H(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \leq \alpha\, S(x, x, y), \qquad (*)$$

for all $x, y \in \Omega$.

Example 2.12. Let (Ω, S) be an S-metric space and let $\Omega = \mathbb{R}$. Define $S\colon \Omega^3 \to [0, +\infty)$ by $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Let $\mathcal{T}\colon \Omega \to CB(\Omega)$ be defined by $\mathcal{T}(x) = \Big[0, \frac{x}{3}\Big]$ for all $x \in \Omega$.

For any $x, y \in \Omega$, and let $x < y$, we have

$$\begin{aligned}
S_H(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) &= S_H\Big(\Big[0, \frac{x}{3}\Big], \Big[0, \frac{x}{3}\Big], \Big[0, \frac{y}{3}\Big]\Big) \\
&= \max\Big\{ \sup_{u \in \left[0, \frac{x}{3}\right]} S\Big(u, u, \Big[0, \frac{y}{3}\Big]\Big), \sup_{v \in \left[0, \frac{y}{3}\right]} S\Big(v, v, \Big[0, \frac{x}{3}\Big]\Big)\Big\} \\
&= \max\Big\{ \sup_{u \in \left[0, \frac{x}{3}\right]} 2\Big|u - \Big[0, \frac{y}{3}\Big]\Big|, \sup_{v \in \left[0, \frac{y}{3}\right]} 2\Big|v - \Big[0, \frac{x}{3}\Big]\Big|\Big\} \\
&= 2\max\Big\{0, \Big|\frac{x}{3} - \frac{y}{3}\Big|\Big\} \\
&= 2\Big|\frac{x}{3} - \frac{y}{3}\Big| = \frac{1}{3}(2|x - y|) \\
&< \frac{1}{2}(2|x - y|) = \frac{1}{2}S(x, x, y) \\
&= \alpha\, S(x, x, y).
\end{aligned}$$

Since there exists $\alpha = \frac{1}{2} \in [0, 1)$ such that

$$S_H(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) < \alpha\, S(x, x, y),$$

it follows that $\mathcal{T}$ is a multivalued contraction mapping.

Theorem 2.13. *([10], Theorem 3.1) Let (Ω, S) be an S-metric space. If $\mathcal{T}\colon \Omega \to CB(\Omega)$ is a multi-valued contraction on Ω, then $\mathcal{T}$ has a fixed point.*

Definition 2.10. ([19]) Let (Ω, S) be an S-metric space, $\mathcal{T}\colon \Omega \to \Omega$ be a mapping and $\zeta \in \mathcal{Z}$. Then $\mathcal{T}$ is called a $\mathcal{Z}_S$-contraction with respect to ζ if the following condition is satisfied

$$\zeta\Big(S(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z), S(x, y, z)\Big) \geq 0, \tag{2.3}$$

for all $x, y, z \in \Omega$.

Remark 2.1. ([19]) Every $\mathcal{Z}_S$-contraction is a contraction and therefore it is continuous (see [34]).

Lemma 2.14. *([19], Lemma 3) Let (Ω, S) be an S-metric space. If $\mathcal{T}\colon \Omega \to \Omega$ be a $\mathcal{Z}_S$-contraction with respect to $\zeta \in \mathcal{Z}$ and $\mathcal{T}$ has a fixed point, then the fixed point is unique.*

Definition 2.11. ([23]) Let (Ω, d) be a metric space, $\mathcal{T}\colon \Omega \to \Omega$ a mapping and $\zeta \in \mathcal{Z}$. Then $\mathcal{T}$ is called a generalized $\mathcal{Z}$-contraction with respect to ζ if

$$\zeta\Big(d(\mathcal{T}x, \mathcal{T}y), M(x, y)\Big) \geq 0, \tag{2.4}$$

for all $x, y \in \Omega$, where

$$M(x, y) = \max\Big\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{d(x, \mathcal{T}y) + d(y, \mathcal{T}x)}{2}\Big\}.$$

On the other hand, *Padcharoen et al.* [25] presented the notion of generalized Suzuki type $\mathcal{Z}$-contraction on a metric space as follows.

Definition 2.12. ([25]) Let (Ω, d) be a metric space, $\mathcal{T}\colon \Omega \to \Omega$ a mapping and $\zeta \in \mathcal{Z}$. Then $\mathcal{T}$ is called a generalized Suzuki type $\mathcal{Z}$-contraction with respect to ζ if

$$\frac{1}{2}d(x, \mathcal{T}x) < d(x, y) \Rightarrow \zeta\Big(d(\mathcal{T}x, \mathcal{T}y), M(x, y)\Big) \geq 0, \tag{2.5}$$

for all distinct $x, y \in \Omega$, where

$$M(x, y) = \max\Big\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{d(x, \mathcal{T}y) + d(y, \mathcal{T}x)}{2}\Big\}.$$

Motivated and inspired by Definition 2.11 and Definition 2.12, we introduce the concept of generalized multi-valued $\mathcal{Z}$-contraction and generalized multi-valued Suzuki type $\mathcal{Z}$-contraction for a pair of mappings in S-metric space.

Definition 2.13. Let (Ω, S) be an S-metric space and $F, G\colon \Omega \to CB(\Omega)$ be mappings. Then the pair (F, G) is said to be a generalized multi-valued $\mathcal{Z}$-contraction for a pair of mappings with respect to $\zeta \in \mathcal{Z}$ if

$$\zeta\Big(S_H(Fx, Fx, Gy), \mathcal{M}_S(x, x, y)\Big) \geq 0, \tag{2.6}$$

for all $x, y \in \Omega$, where

$$\begin{aligned}\mathcal{M}_S(x, x, y) \;=\; \max\Big\{&S(x, x, y), S(x, x, Fx), S(y, y, Gy),\\ &\frac{S(x, x, y) + S(y, y, Fx)}{2}\Big\}.\end{aligned}$$

Definition 2.14. Let (Ω, S) be an S-metric space and $F, G\colon \Omega \to CB(\Omega)$ be mappings. Then the pair (F, G) is said to be a generalized multi-valued Suzuki type $\mathcal{Z}$-contraction for pair of mappings with respect to $\zeta \in \mathcal{Z}$ if

$$\begin{aligned}&\frac{1}{3}\min\Big\{S(x, x, Fx), S(y, y, Gy)\Big\} < S(x, x, y)\\ &\Rightarrow \zeta\Big(S_H(Fx, Fx, Gy), \mathcal{M}_S(x, x, y)\Big) \geq 0,\end{aligned} \tag{2.7}$$

for all $x, y \in \Omega$, where

$$\begin{aligned}\mathcal{M}_S(x, x, y) \;=\; \max\Big\{&S(x, x, y), S(x, x, Fx), S(y, y, Gy),\\ &\frac{S(x, x, y) + S(y, y, Fx)}{2}\Big\}.\end{aligned}$$

3. Main Results

In this section, we prove some common fixed point theorems for generalized multivalued Suzuki type $\mathcal{Z}$-contraction for a pair of mappings with respect to ζ in the framework of S-metric spaces.

Theorem 3.1. *Let (Ω, S) be a complete S-metric space and $F, G\colon \Omega \to CB(\Omega)$ be a pair of generalized multivalued Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ as defined in (2.7). Then F and G have a common fixed point in Ω.*

Proof. Let $x_0 \in \Omega$ be an arbitrary point. Choose $x_1 \in Fx_0$. Then by the definition of the Hausdorff S-metric, there exists $x_2 \in Gx_1$ such that

$$0 < S(x_1, x_1, x_2) = S(x_1, x_1, Gx_1) \leq S_H(Fx_0, Fx_0, Gx_1). \tag{3.1}$$

Assume that $S(x_0, x_0, Fx_0) > 0$ and $S(x_1, x_1, Gx_1) > 0$, then

$$\frac{1}{3} \min \Big\{ S(x_0, x_0, Fx_0), S(x_1, x_1, Gx_1) \Big\} < S(x_0, x_0, x_1). \tag{3.2}$$

Therefore from (2.7), we have

$$\begin{aligned} 0 &\leq \zeta\Big(S_H(Fx_0, Fx_0, Gx_1), \mathcal{M}_S(x_0, x_0, x_1)\Big) \\ &< \mathcal{M}_S(x_0, x_0, x_1) - S_H(Fx_0, Fx_0, Gx_1). \end{aligned}$$

Consequently, we get

$$\begin{aligned} S(x_1, x_1, x_2) &\leq S_H(Fx_0, Fx_0, Gx_1) \\ &< \mathcal{M}_S(x_0, x_0, x_1), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mathcal{M}_S(x_0, x_0, x_1) &= \max \Big\{ S(x_0, x_0, x_1), S(x_0, x_0, Fx_0), S(x_1, x_1, Gx_1), \\ &\qquad \frac{S(x_0, x_0, x_1) + S(x_1, x_1, Fx_0)}{2} \Big\} \\ &= \max \Big\{ S(x_0, x_0, x_1), S(x_0, x_0, x_1), S(x_1, x_1, x_2), \\ &\qquad \frac{S(x_0, x_0, x_1) + S(x_1, x_1, x_1)}{2} \Big\} \\ &= \max \Big\{ S(x_0, x_0, x_1), S(x_1, x_1, x_2) \Big\}. \end{aligned}$$

Suppose that $\max \big\{ S(x_0, x_0, x_1), S(x_1, x_1, x_2) \big\} = S(x_1, x_1, x_2)$, then (3.3) becomes

$$S(x_1, x_1, x_2) \leq S_H(Fx_0, Fx_0, Gx_1) < S(x_1, x_1, x_2),$$

which is a contradiction. Thus we conclude that

$$\max \big\{ S(x_0, x_0, x_1), S(x_1, x_1, x_2) \big\} = S(x_0, x_0, x_1).$$

By (3.3), we obtain

$$S(x_1, x_1, x_2) < S(x_0, x_0, x_1).$$

Similarly, for $x_2 \in Gx_1$ and $x_3 \in Fx_2$, we have

$$S(x_2, x_2, x_3) \leq S_H(Gx_1, Gx_1, Fx_2) < S(x_1, x_1, x_2).$$

This implies

$$S(x_2, x_2, x_3) \leq S(x_1, x_1, x_2).$$

By continuing the same process as above, we construct a sequence $\{x_n\}$ in Ω such that $x_{2n+1} \in Fx_{2n}$ and $x_{2n+2} \in Gx_{2n+1}$, $n = 0, 1, 2, \ldots$ such that

$$\begin{aligned} 0 < S(x_{2n+1}, x_{2n+1}, x_{2n+2}) &= S(x_{2n+1}, x_{2n+1}, Gx_{2n+1}) \\ &\leq S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}), \end{aligned}$$

and

$$\frac{1}{3}\min\Big\{S(x_{2n},x_{2n},Fx_{2n}),S(x_{2n+1},x_{2n+1},Gx_{2n+1})\Big\}$$
$$< S(x_{2n},x_{2n},x_{2n+1}).$$

Hence from (2.7), we have

$$\begin{aligned} 0 &\leq \zeta\Big(S_H(Fx_{2n},Fx_{2n},Gx_{2n+1}),\mathcal{M}_S(x_{2n},x_{2n},x_{2n+1})\Big) \\ &< \mathcal{M}_S(x_{2n},x_{2n},x_{2n+1})-S_H(Fx_{2n},Fx_{2n},Gx_{2n+1}). \end{aligned}$$

Consequently, we get

$$\begin{aligned} S(x_{2n+1},x_{2n+1},x_{2n+2}) &\leq S_H(Fx_{2n},Fx_{2n},Gx_{2n+1}) \\ &< \mathcal{M}_S(x_{2n},x_{2n},x_{2n+1}), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \mathcal{M}_S(x_{2n},x_{2n},x_{2n+1}) &= \max\Big\{S(x_{2n},x_{2n},x_{2n+1}),S(x_{2n},x_{2n},Fx_{2n}), \\ &\qquad S(x_{2n+1},x_{2n+1},Gx_{2n+1}), \\ &\qquad \frac{S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n+1},x_{2n+1},Fx_{2n})}{2}\Big\} \\ &= \max\Big\{S(x_{2n},x_{2n},x_{2n+1}),S(x_{2n},x_{2n},x_{2n+1}), \\ &\qquad S(x_{2n+1},x_{2n+1},x_{2n+2}), \\ &\qquad \frac{S(x_{2n},x_{2n},x_{2n+1})+S(x_{2n+1},x_{2n+1},x_{2n+1})}{2}\Big\} \\ &= \max\Big\{S(x_{2n},x_{2n},x_{2n+1}),S(x_{2n+1},x_{2n+1},x_{2n+2})\Big\}. \end{aligned}$$

Suppose that $\max\Big\{S(x_{2n},x_{2n},x_{2n+1}),S(x_{2n+1},x_{2n+1},x_{2n+2})\Big\}=S(x_{2n+1},x_{2n+1},x_{2n+2})$, then from (3.4), we have

$$S(x_{2n+1},x_{2n+1},x_{2n+2})\leq S_H(Fx_{2n},Fx_{2n},Gx_{2n+1})<S(x_{2n+1},x_{2n+1},x_{2n+2}),$$

which is a contradiction. So we conclude that

$$\max\Big\{S(x_{2n},x_{2n},x_{2n+1}),S(x_{2n+1},x_{2n+1},x_{2n+2})\Big\}=S(x_{2n},x_{2n},x_{2n+1}).$$

Then from (3.4), we obtain

$$S(x_{2n+1},x_{2n+1},x_{2n+2})\leq S_H(Fx_{2n},Fx_{2n},Gx_{2n+1})<S(x_{2n},x_{2n},x_{2n+1}).$$

This implies that

$$S(x_{2n+1},x_{2n+1},x_{2n+2})<S(x_{2n},x_{2n},x_{2n+1}). \tag{3.5}$$

Hence $S(x_{n+1},x_{n+1},x_{n+2})<S(x_n,x_n,x_{n+1})$ for all n.

Therefore $\{S(x_n,x_n,x_{n+1})\}$ is a strictly decreasing sequence of non-negative real numbers. Thus there exists $d\geq 0$ such that

$$\lim_{n\to\infty}S(x_n,x_n,x_{n+1})=d.$$

We shall prove that $d = 0$. Suppose on the contrary that $d > 0$. Now by using condition (ζ_3) with $t_n = S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1})$ and $s_n = S(x_{2n}, x_{2n}, x_{2n+1})$, we have

$$0 \leq \limsup_{n\to\infty} \zeta\big(S_H(Fx_{2n}, Fx_{2n}, Gx_{2n+1}), S(x_{2n}, x_{2n}, x_{2n+1})\big) < 0,$$

which is a contradiction. Thus, we conclude that $d = 0$, that is,

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{3.6}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence. We suppose that $\{x_n\}$ is not a Cauchy sequence.

Then using equations (2.7), (3.6) and Lemma 2.9, we can choose a positive integer $n_0 \geq 1$ such that

$$\frac{1}{3}\min\Big\{S(x_{n(k)}, x_{n(k)}, Fx_{n(k)}), S(x_{m(k)}, x_{m(k)}, Gx_{m(k)})\Big\}$$
$$< \frac{\varepsilon}{2} < S(x_{n(k)}, x_{n(k)}, x_{m(k)}),$$

and

$$\begin{aligned} 0 &\leq \zeta\big(S_H(S(Fx_{n(k)}, Fx_{n(k)}, Gx_{m(k)})), \mathcal{M}_S(x_{n(k)}, x_{n(k)}, x_{m(k)})\big) \\ &< \mathcal{M}_S(x_{n(k)}, x_{n(k)}, x_{m(k)}) - S_H(S(Fx_{n(k)}, Fx_{n(k)}, Gx_{m(k)})). \end{aligned}$$

Consequently, we get

$$\begin{aligned} S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)+1}) &\leq S_H(S(Fx_{n(k)}, Fx_{n(k)}, Gx_{m(k)})) \\ &< \mathcal{M}_S(x_{n(k)}, x_{n(k)}, x_{m(k)}), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \mathcal{M}_S(x_{n(k)}, x_{n(k)}, x_{m(k)}) &= \max\Big\{S(x_{n(k)}, x_{n(k)}, x_{m(k)}), S(x_{n(k)}, x_{n(k)}, Fx_{n(k)}), \\ &\quad S(x_{m(k)}, x_{m(k)}, Gx_{m(k)}), \\ &\quad \frac{S(x_{n(k)}, x_{n(k)}, x_{m(k)}) + S(x_{m(k)}, x_{m(k)}, Fx_{n(k)})}{2}\Big\} \\ &= \max\Big\{S(x_{n(k)}, x_{n(k)}, x_{m(k)}), S(x_{n(k)}, x_{n(k)}, x_{n(k)+1}), \\ &\quad S(x_{m(k)}, x_{m(k)}, x_{m(k)+1}), \\ &\quad \frac{S(x_{n(k)}, x_{n(k)}, x_{m(k)}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)+1})}{2}\Big\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.6), Lemma 2.5 and Lemma 2.9, we get

$$\lim_{k\to\infty} \mathcal{M}_S(x_{n(k)}, x_{n(k)}, x_{m(k)}) = \max\{\varepsilon, 0, 0, \varepsilon\} = \varepsilon. \tag{3.8}$$

Now by using condition (ζ_3) with $t_n = S_H(Fx_{n(k)}, Fx_{n(k)}, Gx_{m(k)})$ and $s_n = \mathcal{M}_S(x_{n(k)}, x_{n(k)}, x_{m(k)})$, we have

$$0 \leq \limsup_{n\to\infty} \zeta\big(S_H(Fx_{n(k)}, Fx_{n(k)}, Gx_{m(k)}), \mathcal{M}_S(x_{n(k)}, x_{n(k)}, x_{m(k)})\big) < 0,$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in Ω. Since Ω is complete, we can ensure that $\{x_n\}$ converges to some $x^* \in \Omega$, that is,

$$\lim_{n\to\infty} S(x_n, x_n, x^*) = 0,$$

and so

$$\lim_{n\to\infty} S(x_n, x_n, x^*) = \lim_{n\to\infty} S(x_{2n}, x_{2n}, x^*) = \lim_{n\to\infty} S(x_{2n+1}, x_{2n+1}, x^*) = 0. \tag{3.9}$$

Now, we claim that

$$\frac{1}{3}\min\Big\{S(x_n, x_n, Fx_n), S(x^*, x^*, Gx^*)\Big\} < S(x_n, x_n, x^*),$$

or

$$\frac{1}{3}\min\Big\{S(x^*, x^*, Fx^*), S(x_{n+1}, x_{n+1}, Gx_{n+1})\Big\} < S(x^*, x^*, x_{n+1}), \tag{3.10}$$

for all $n \in \mathbb{N}$. Suppose that it is not the case. Then there exists $m \in \mathbb{N}$ such that

$$\frac{1}{3}\min\Big\{S(x_m, x_m, Fx_m), S(x^*, x^*, Gx^*)\Big\} \geq S(x_m, x_m, x^*), \tag{3.11}$$

and

$$\frac{1}{3}\min\Big\{S(x^*, x^*, Fx^*), S(x_{m+1}, x_{m+1}, Gx_{m+1})\Big\} \geq S(x^*, x^*, x_{m+1}). \tag{3.12}$$

Therefore

$$\begin{aligned} 3S(x_m, x_m, x^*) &\leq \min\Big\{S(x_m, x_m, Fx_m), S(x^*, x^*, Gx^*)\Big\} \\ &\leq \min\Big\{2S(x_m, x_m, x^*) + S(x^*, x^*, Fx_m), S(x^*, x^*, Gx^*)\Big\} \\ &\leq 2S(x_m, x_m, x^*) + S(x^*, x^*, Fx_m) \\ &\leq 2S(x_m, x_m, x^*) + S(x^*, x^*, x_{m+1}), \end{aligned}$$

which implies by Lemma 2.5, we have

$$S(x_m, x_m, x^*) \leq S(x_{m+1}, x_{m+1}, x^*). \tag{3.13}$$

From (3.12) and (3.13), we have

$$\begin{aligned} S(x_m, x_m, x^*) &\leq S(x_{m+1}, x_{m+1}, x^*) \\ &\leq \frac{1}{3}\min\Big\{S(x^*, x^*, Fx^*), S(x_{m+1}, x_{m+1}, Gx_{m+1})\Big\}. \end{aligned} \tag{3.14}$$

Since

$$\frac{1}{3}\min\Big\{S(x_m, x_m, Fx_m), S(x^*, x^*, Gx^*)\Big\} < S(x_m, x_m, x_{m+1}),$$

from (2.7), we have

$$\begin{aligned} 0 &\leq \zeta\big(S_H(Fx_m, Fx_m, Gx_{m+1}), \mathcal{M}_S(x_m, x_m, x_{m+1})\big) \\ &< \mathcal{M}_S(x_m, x_m, x_{m+1}) - S_H(Fx_m, Fx_m, Gx_{m+1}). \end{aligned}$$

Consequently, we get

$$\begin{aligned} S(x_{m+1}, x_{m+1}, x_{m+2}) &\leq S_H(Fx_m, Fx_m, Gx_{m+1}) \\ &< \mathcal{M}_S(x_m, x_m, x_{m+1}), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \mathcal{M}_S(x_m, x_m, x_{m+1}) &= \max\Big\{S(x_m, x_m, x_{m+1}), S(x_m, x_m, Fx_m), \\ &\quad S(x_{m+1}, x_{m+1}, Gx_{m+1}), \\ &\quad \frac{S(x_m, x_m, x_{m+1}) + S(x_{m+1}, x_{m+1}, Fx_m)}{2}\Big\} \\ &= \max\Big\{S(x_m, x_m, x_{m+1}), S(x_m, x_m, x_{m+1}), \\ &\quad S(x_{m+1}, x_{m+1}, x_{m+2}), \\ &\quad \frac{S(x_m, x_m, x_{m+1}) + S(x_{m+1}, x_{m+1}, x_{m+1})}{2}\Big\} \\ &= \max\Big\{S(x_m, x_m, x_{m+1}), S(x_{m+1}, x_{m+1}, x_{m+2})\Big\}. \end{aligned}$$

Suppose that $\max\Big\{S(x_m, x_m, x_{m+1}), S(x_{m+1}, x_{m+1}, x_{m+2})\Big\} = S(x_{m+1}, x_{m+1}, x_{m+2})$, then from (3.15), we obtain

$$S(x_{m+1}, x_{m+1}, x_{m+2}) \leq S_H(Fx_m, Fx_m, Gx_{m+1}) < S(x_{m+1}, x_{m+1}, x_{m+2}),$$

which is a contradiction. Thus, we conclude that

$$\max\Big\{S(x_m, x_m, x_{m+1}), S(x_{m+1}, x_{m+1}, x_{m+2})\Big\} = S(x_m, x_m, x_{m+1}).$$

Hence from (3.15), we obtain

$$S(x_{m+1}, x_{m+1}, x_{m+2}) < S(x_m, x_m, x_{m+1}). \tag{3.16}$$

From (3.14), (3.16) and Lemma 2.5, we get

$$\begin{aligned} S(x_{m+1}, x_{m+1}, x_{m+2}) &\leq S(x_m, x_m, x_{m+1}) \\ &\leq 2S(x_m, x_m, x^*) + S(x_{m+1}, x_{m+1}, x^*) \\ &= 2S(x_m, x_m, x^*) + S(x^*, x^*, x_{m+1}) \\ &\leq \frac{2}{3}\min\Big\{S(x^*, x^*, Fx^*), S(x_{m+1}, x_{m+1}, Gx_{m+1})\Big\} \\ &\quad + \frac{1}{3}\min\Big\{S(x^*, x^*, Fx^*), S(x_{m+1}, x_{m+1}, Gx_{m+1})\Big\} \\ &= \min\Big\{S(x^*, x^*, Fx^*), S(x_{m+1}, x_{m+1}, Gx_{m+1})\Big\} \\ &\leq S(x_{m+1}, x_{m+1}, x_{m+2}), \end{aligned}$$

which is a contradiction. Hence (3.10) holds, that is, for every $n \geq 2$

$$\frac{1}{3}\min\Big\{S(x_n, x_n, Fx_n), S(x^*, x^*, Gx^*)\Big\} < S(x_n, x_n, x^*)$$

holds. Hence from (2.7), it follows that for every $n \geq 2$,

$$\begin{aligned} 0 &\leq \zeta\big(S_H(Fx_n, Fx_n, Gx^*), \mathcal{M}_S(x_n, x_n, x^*)\big) \\ &< \mathcal{M}_S(x_n, x_n, x^*) - S_H(Fx_n, Fx_n, Gx^*). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} S(x_{n+1}, x_{n+1}, Gx^*) &\leq S_H(Fx_n, Fx_n, Gx^*) \\ &< \mathcal{M}_S(x_n, x_n, x^*), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} \mathcal{M}_S(x_n, x_n, x^*) &= \max\Big\{S(x_n, x_n, x^*), S(x_n, x_n, Fx_n), S(x^*, x^*, Gx^*), \\ &\qquad \frac{S(x_n, x_n, x^*) + S(x^*, x^*, Fx_n)}{2}\Big\} \\ &= \max\Big\{S(x_n, x_n, x^*), S(x_n, x_n, x_{n+1}), S(x^*, x^*, Gx^*), \\ &\qquad \frac{S(x_n, x_n, x^*) + S(x^*, x^*, x_{n+1})}{2}\Big\}. \end{aligned}$$

Letting $n \to \infty$ and by using (3.6), (3.9), condition (a) and Lemma 2.7, we get

$$\begin{aligned} \lim_{n\to\infty} \mathcal{M}_S(x_n, x_n, x^*) &= \max\Big\{0, 0, S(x^*, x^*, Gx^*), 0\Big\} \\ &= S(x^*, x^*, Gx^*). \end{aligned} \tag{3.18}$$

Now, we prove that $x^* \in Gx^*$. Suppose on the contrary that $S(x^*, x^*, Gx^*) > 0$. Passing to the limit as $n \to \infty$ in (3.17) and using (3.18), we obtain

$$\begin{aligned} S(x^*, x^*, Gx^*) &= \lim_{n\to\infty} S(x_{n+1}, x_{n+1}, Gx^*) \\ &\leq \lim_{n\to\infty} S_H(Fx_n, Fx_n, Gx^*) \\ &\leq \lim_{n\to\infty} \mathcal{M}_S(x_n, x_n, x^*) = S(x^*, x^*, Gx^*), \end{aligned}$$

which is a contradiction. Therefore $x^* \in Gx^*$. Likewise, we can prove that $x^* \in Fx^*$. Thus, F and G have a common fixed point. This completes the proof. □

Example 3.2. Let $\Omega = \{0, 1, 2\}$ and $S\colon \Omega^3 \to [0, +\infty)$ be defined by

$$\begin{aligned} S(0,0,1) = 1,\ S(0,0,2) = 2,\ S(1,1,2) = 3, \text{ and } S(0,0,0) &= S(1,1,1) \\ &= S(2,2,2) = 0. \end{aligned}$$

Then (Ω, S) is a complete S-metric space.

Define the mappings $F, G\colon \Omega \to CB(\Omega)$ by

$$F(x) = \begin{cases} \{0\}, & \text{if } x \in \{0,1\}, \\ \{0,1\}, & \text{if } x = 2, \end{cases}$$

and

$$G(x) = \begin{cases} \{0\}, & \text{if } x \in \{0,1\}, \\ \{1\}, & \text{if } x = 2. \end{cases}$$

Let $\zeta\colon [0,+\infty)\times[0,+\infty)\to\mathbb{R}$ be defined by $\zeta(t,s)=\frac{9}{10}s-t$ for all $s,t\in[0,+\infty)$. Now verify inequality (2.7) for all $x,y\in\Omega$ with $F(x)\neq G(x)$. Note that for all $x,y\in\{0,1,2\}$ with $F(x)\neq G(x)$ the inequality

$$\frac{1}{3}\min\Big\{S(x,x,Fx),S(y,y,Gy)\Big\}<S(x,x,y)$$

gives $S(x,x,y)\in\{S(0,0,2),S(2,2,0),S(1,1,2),S(2,2,1)\}$. Then from (2.7), we have

$$\zeta\big(S_H(Fx,Fx,Gy),\mathcal{M}_S(x,x,y)\big)=\frac{9}{10}\mathcal{M}_S(x,x,y)-S_H(Fx,Fx,Gy)\geq 0.$$

This implies that

$$S_H(Fx,Fx,Gy)\leq\frac{9}{10}\mathcal{M}_S(x,x,y).$$

Case (i): for $x=0$, $y=2$;

$$S_H(F0,F0,G2)=S_H\big(\{0\},\{0\},\{1\}\big)=1\leq\frac{9}{10}\mathcal{M}_S(0,0,2)=\frac{9}{5}.$$

Case (ii): for $x=2$, $y=0$;

$$S_H(F2,F2,G0)=S_H\big(\{0,1\},\{0,1\},\{0\}\big)=1\leq\frac{9}{10}\mathcal{M}_S(2,2,0)=\frac{9}{5}.$$

Case (iii): for $x=1$, $y=2$;

$$S_H(F1,F1,G2)=S_H\big(\{0\},\{0\},\{1\}\big)=1\leq\frac{9}{10}\mathcal{M}_S(1,1,2)=\frac{27}{10}.$$

Case (iv): for $x=2$, $y=1$;

$$S_H(F2,F2,G1)=S_H\big(\{0,1\},\{0,1\},\{0\}\big)=1\leq\frac{9}{10}\mathcal{M}_S(2,2,1)=\frac{27}{10}.$$

Thus all the assumptions of Theorem 3.1 are satisfied. Hence 0 is a common fixed point of F and G.

Now, we present some auxiliary results obtained from Theorem 3.1.

Corollary 3.3. *Let (Ω,S) be a complete S-metric space and $F,G\colon\Omega\to CB(\Omega)$ be a pair of generalized multivalued $\mathcal{Z}$-contraction with respect to ζ (see, equation (2.6)). Then F and G have a common fixed point $x^*\in\Omega$.*

Corollary 3.4. *Let (Ω,S) be a complete S-metric space and $F\colon\Omega\to CB(\Omega)$ be a generalized multivalued Suzuki type $\mathcal{Z}$-contraction with respect to ζ, that is,*

$$\frac{1}{3}S(x,x,Fx)<S(x,x,y)\Rightarrow\zeta\Big(S_H(Fx,Fx,Fy),\mathcal{M}_S(x,x,y)\Big)\geq 0, \tag{3.19}$$

for all $x,y\in\Omega$ with $x\neq y$, where

$$\mathcal{M}_S(x,x,y)=\max\Big\{S(x,x,y),S(x,x,Fx),S(y,y,Fy),\frac{S(x,x,y)+S(y,y,Fx)}{2}\Big\}.$$

Then F has a fixed point $x^*\in\Omega$ and for $x\in\Omega$ the sequence $\{F^nx\}$ converges to x^*.

Proof. It follows from Theorem 3.1 by taking $F=G$. □

Example 3.5. Let $\Omega = \{0,1,2\}$ and $S\colon \Omega^3 \to [0,+\infty)$ be defined by

$$S(0,0,1) = 1,\ S(0,0,2) = 2,\ S(1,1,2) = 2, \text{ and } S(0,0,0) = S(1,1,1) = S(2,2,2) = 0.$$

Then (Ω, S) is a complete S-metric space.

Define the mapping $F\colon \Omega \to CB(\Omega)$ by

$$F0 = F1 = \{0\},\ F2 = \{0,1\}.$$

Note that for all distinct $x, y \in \Omega$ the inequality $\frac{1}{3}S(x,x,Fx) < S(x,x,y)$. Hence from (3.19), we shall prove that

$$\zeta\big(S_H(Fx,Fx,Fy), \mathcal{M}_S(x,x,y)\big) \geq 0.$$

We, now define $\zeta\colon [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ by $\zeta(t,s) = \frac{7}{8}s - t$ for all $s,t \in [0,+\infty)$. Thus, we have

$$\zeta\big(S_H(Fx,Fx,Fy), \mathcal{M}_S(x,x,y)\big) = \frac{7}{8}\mathcal{M}_S(x,x,y) - S_H(Fx,Fx,Fy) \geq 0.$$

Therefore, this implies that

$$S_H(Fx,Fx,Fy) \leq \frac{7}{8}\mathcal{M}_S(x,x,y).$$

Case (1): $x, y \in \{0,1\}$, we have

$$S_H(Fx,Fx,Fy) = S(0,0,0) = 0 \leq \frac{7}{8}\mathcal{M}_S(x,x,y).$$

Case (2): for $x \in \{0,1\}$, $y = 2$, we have

$$S_H(Fx,Fx,Fy) = S_H\big(\{0\},\{0\},\{0,1\}\big) = \max\{0,1\} = 1 \leq \frac{7}{8}\mathcal{M}_S(x,x,y) = 1.75.$$

Thus all the assumptions of Corollary 3.4 are satisfied and 0 is the unique fixed point of F.

Corollary 3.6. *Let (Ω, S) be a complete S-metric space and $F, G\colon \Omega \to \Omega$ be a pair of generalized single valued Suzuki type $\mathcal{Z}$-contractions with respect to ζ, that is,*

$$\begin{aligned} &\frac{1}{3}\min\Big\{S(x,x,Fx), S(y,y,Gy)\Big\} < S(x,x,y) \\ &\Rightarrow \zeta\Big(S(Fx,Fx,Gy), \mathcal{M}_S(x,x,y)\Big) \geq 0, \end{aligned} \tag{3.20}$$

for all $x, y \in \Omega$ with $Fx \neq Gy$, where

$$\mathcal{M}_S(x,x,y) = \max\Big\{S(x,x,y), S(x,x,Fx), S(y,y,Gy), \frac{S(x,x,y) + S(y,y,Fx)}{2}\Big\}.$$

Then F and G have a unique common fixed point $x^* \in \Omega$ and for $x \in \Omega$ the sequence $\{F^n x\}$ converges to x^*.

Proof. It can be proved easily by taking F and G as single valued mappings in Theorem 3.1. Uniqueness of the common fixed point is obvious. □

Corollary 3.7. *Let (Ω, S) be a complete S-metric space and $F, G\colon \Omega \to \Omega$ be a pair of generalized single valued $\mathcal{Z}$-contraction with respect to ζ, that is,*

$$\zeta\Big(S(Fx, Fx, Gy), \mathcal{M}_S(x, x, y)\Big) \geq 0, \tag{3.21}$$

for all $x, y \in \Omega$, where

$$\begin{aligned}\mathcal{M}_S(x, x, y) \;\; = \;\; \max\Big\{&S(x, x, y), S(x, x, Fx), S(y, y, Gy), \\ &\frac{S(x, x, y) + S(y, y, Fx)}{2}\Big\}.\end{aligned}$$

Then F and G have a unique common fixed point $x^* \in \Omega$ and for $x \in \Omega$ the sequence $\{F^n x\}$ converges to x^*.

Remark 3.1. Our results extend and generalise the corresponding results of *Antal and Gairola* [1] and *Padcharoen et al.* [25] from complete metric space to complete S-metric space.

If we take $\zeta(t, s) = \lambda\, s - t$ for all $t, s \in [0, \infty)$ with $\lambda \in [0, 1)$ in Corollary 3.3, then we have the following result.

Corollary 3.8. *Let (Ω, S) be a complete S-metric space and $F, G\colon \Omega \to CB(\Omega)$ be a pair of generalized multivalued contraction such that*

$$S_H(Fx, Fx, Gy) \leq \lambda\, \mathcal{M}_S(x, x, y), \tag{3.22}$$

for all $x, y \in \Omega$ and $\lambda \in [0, 1)$ with $Fx \neq Gy$, where

$$\begin{aligned}\mathcal{M}_S(x, x, y) \;\; = \;\; \max\Big\{&S(x, x, y), S(x, x, Fx), S(y, y, Gy), \\ &\frac{S(x, x, y) + S(y, y, Fx)}{2}\Big\}.\end{aligned}$$

Then F and G have a common fixed point $x^* \in \Omega$.

Remark 3.2. Corollary 3.8 extends and generalizes Theorem 3.1 of *Gairola and Khantwal* [10] for pair of multi-valued mappings.

If we take $\zeta(t, s) = \lambda\, s - t$ for all $t, s \in [0, \infty)$ with $\lambda \in [0, 1)$, $F = G$ and

$$\max\Big\{S(x, x, y), S(x, x, Fx), S(y, y, Fy), \frac{S(x, x, y) + S(y, y, Fx)}{2}\Big\} = S(x, x, y)$$

in Corollary 3.3, then we have the following result.

Corollary 3.9. *([10], Theorem 3.1) Let (Ω, S) be a complete S-metric space and $F\colon \Omega \to CB(\Omega)$ be a generalized multivalued contraction such that*

$$S_H(Fx, Fx, Fy) \leq \lambda\, S(x, x, y), \tag{3.23}$$

for all $x, y \in \Omega$, $x \neq y$ and $\lambda \in [0, 1)$ is a constant. Then F has a fixed point $x^* \in \Omega$.

Example 3.10. Let $\Omega = [0, +\infty)$. Define $S\colon \Omega^3 \to [0, +\infty)$ by $S(x, y, z) = |x - z| + |y - z|$. Then (Ω, S) is a complete S-metric space.

Let $F, G\colon \Omega \to CB(\Omega)$ be defined by $F(x) = \Big[\frac{x}{3}, \frac{x}{6}\Big]$ and $G(y) = \Big[\frac{y}{3}, \frac{y}{6}\Big]$ for all $x, y \in \Omega$. Now, we have

$$
\begin{aligned}
S_H(Fx,Fx,Gy) &= \max\Big\{\sup_{x\in Fx} S(x,x,Gy), \sup_{y\in Gy} S(y,y,Fx)\Big\}\\
&= \max\Big\{\sup_{x\in Fx} 2\Big|x-Gy\Big|, \sup_{y\in Gy} 2\Big|y-Fx\Big|\Big\}\\
&= 2\max\Big\{\sup_{x\in\left[\frac{x}{3},\frac{x}{6}\right]}\Big|x-[\frac{y}{3},\frac{y}{6}]\Big|, \sup_{y\in\left[\frac{y}{3},\frac{y}{6}\right]}\Big|y-[\frac{x}{3},\frac{x}{6}]\Big|\Big\}\\
&= 2\max\Big\{\Big|\frac{x}{3}-\frac{y}{3}\Big|, \Big|\frac{x}{6}-\frac{y}{6}\Big|\Big\}\\
&= \frac{2}{3}\max\Big\{|x-y|, \Big|\frac{x}{2}-\frac{y}{2}\Big|\Big\}\\
&= \frac{2}{3}|x-y| = \frac{1}{3}(2|x-y|) = \frac{1}{3}S(x,x,y)\\
&\leq \frac{1}{2}S(x,x,y) = \lambda\, S(x,x,y)\\
&\leq \lambda\,\mathcal{M}_S(x,x,y),
\end{aligned}
$$

where $\lambda = \frac{1}{2} < 1$. Thus all the assumptions of Corollary 3.8 are satisfied. Hence by application of Corollary 3.8, F and G have a common fixed point.

Example 3.11. Let $\Omega = [0,1]$. Define the function $S\colon \Omega^3 \to [0,+\infty)$ by

$$S(x,y,z) = \frac{|x-z|^2+|y-z|^2}{2}$$

for all $x,y,z\in\Omega$. The condition (a) holds directly. To check condition (b), for all $t\in\Omega$, we have

$$
\begin{aligned}
S(x,x,t)+S(y,y,t)+S(z,z,t) &= |x-t|^2+|y-t|^2+|z-t|^2\\
&= |x-z+z-t|^2+|y-z+z-t|^2+|z-t|^2\\
&\geq (|x-z|^2+|z-t|^2)+(|y-z|^2+|z-t|^2)+|z-t|^2\\
&= 3|z-t|^2+(|x-z|^2+|y-z|^2)\\
&\geq |x-z|^2+|y-z|^2\\
&\geq \frac{|x-z|^2+|y-z|^2}{2} = S(x,y,z).
\end{aligned}
$$

Hence (Ω,S) is an S-MS. Define MV-mappings $F,G\colon\Omega\to CB(\Omega)$ by $F(x) = \left[\frac{x}{3},0\right]$ and $G(y) = \{\frac{y}{3}\}$ for all $x,y\in\Omega$. Now, we examine the condition (3.22). Using Definition 2.8, for all $x,y\in\Omega$, we get

$$
\begin{aligned}
S_H(Fx,Fx,Gy) &= H_S(Fx,Gy)+H_S(Fx,Gy)\\
&= 2\,H_S(Fx,Gy) = 2\max\big\{h_S(Fx,Gy), h_S(Fx,Gy)\big\},
\end{aligned}
$$

where

$$\begin{aligned} h_S(Fx, Gy) &= \sup_{u\in Fx}\inf_{v\in Gy} S(u,u,v) \\ &= \sup_{u\in Fx}\inf_{v\in Gy} \Big(\frac{|u-v|^2+|u-v|^2}{2}\Big) \\ &= \sup_{u\in Fx}\inf_{v\in Gy} \{|u-v|^2\} \\ &= \sup_{u\in Fx}\inf \Big\{\Big|u-\frac{y}{3}\Big|^2\Big\}. \end{aligned}$$

If $u = \frac{x}{3} \in Fx$, then $\inf\Big\{\Big|\frac{x}{3}-\frac{y}{3}\Big|^2\Big\} = 0$. If $u = 0 \in Fx$, then $\inf\Big\{\Big|\frac{x}{3}-\frac{y}{3}\Big|^2\Big\} = 0$. Consequently, $h_S(Fx, Gy) = 0$. Hence

$$S_H(Fx, Fx, Gy) = 2\,\max\{0,0\} = 0 \le \lambda\, S(x,x,y) \le \lambda\, \mathcal{M}_S(x,x,y).$$

Thus, the contractive condition (3.22) is fulfilled with any $\lambda \in [0,1)$. Therefore, all requirements of Corollary 3.8 are satisfied. Hence F and G have a unique common fixed point which is $0 \in \Omega$.

4. Application

In this section, we establish an existence results for the solution of a system of non-linear integral equations.

Consider the following system of nonlinear integral equations

$$\begin{cases} x(t) = \gamma(t) + \displaystyle\int_a^b \mathcal{N}_1(t,s,x(s))ds, \quad t \in [a,b], \\ x(t) = \gamma(t) + \displaystyle\int_a^b \mathcal{N}_2(t,s,x(s))ds, \quad t \in [a,b]. \end{cases} \tag{4.1}$$

Let $\Omega = C([a,b],\mathbb{R})$ represents the set of all continuous real-valued functions on $[a,b]$. Define an S-metric $S\colon \Omega^3 \to \mathbb{R}$ by

$$S(x,y,z) = \|x-z\| + \|y-z\| = \sup_{t\in[a,b]} \big\{|x(t)-z(t)| + |y(t)-z(t)|\big\},$$

for all $x, y, z \in \Omega$.

Then (Ω, S) is a complete S-metric space. Regarding the existence of the solution of the system of nonlinear integral equations (4.1), we prove the following theorem.

Theorem 4.1. *Assume that the assumptions below hold:*

(Υ_1) $\mathcal{N}_j\colon [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ *for* $j = 1,2$ *and* $\gamma\colon [a,b] \to \mathbb{R}$ *are continuous functions.*

(Υ_2) *Suppose there exists* $\lambda \in [0,1)$ *and* $\theta\colon \Omega\times\Omega \to [0,\infty)$ *such that for all* $x, y \in \Omega$ *and* $s \in [a,b]$*, we have*

$$0 \le |\mathcal{N}_1(t,s,x(s)) - \mathcal{N}_2(t,s,y(s))| \le \theta(t,s)\,|x(s)-y(s)|. \tag{4.2}$$

(Υ_3)

$$\sup_{t\in[a,b]} \int_a^b |\theta(t,s)|ds \le \lambda, \ \textit{where}\ 0 \le \lambda < 1.$$

Then, the integral equations (4.1) have a unique common solution x^* in Ω.

Proof. For $x \in \Omega$ and $t \in [a, b]$, define the mappings $F, G\colon C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$\begin{cases} F(x(t)) = \gamma(t) + \displaystyle\int_a^b \mathcal{N}_1(t, r, x(r))dr, & t \in [a,b], \\ G(x(t)) = \gamma(t) + \displaystyle\int_a^b \mathcal{N}_2(t, r, x(r))dr, & t \in [a,b]. \end{cases}$$

Thus, by condition (4.2) of Theorem 4.1, we have

$$\begin{aligned} |F(x(t)) - G(y(t))| &\leq \int_a^b |\mathcal{N}_1(t, s, x(s)) - \mathcal{N}_2(t, s, y(s))| ds \\ &\leq \int_a^b \theta(t, s)(|x(s) - y(s)|) ds \\ &\leq \lambda \, \|x - y\|, \end{aligned}$$

or equivalently,

$$2|F(x(t)) - G(y(t))| \leq 2\lambda \, \|x - y\|.$$

We deduce that for all $x, y \in \Omega$ and $t \in [a,b]$

$$S(Fx, Fx, Gy) \leq \lambda \, S(x, x, y) \leq \lambda \, \mathcal{M}_S(x, x, y).$$

Thus all the hypotheses of Corollary 3.8 are satisfied. Then by application of Corollary 3.8, the system of nonlinear integral equations (4.1) have a unique common solution $x^* \in \Omega$. □

5. Conclusion

In this work, we introduce the notion of generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of mappings and establish common fixed point theorems for such mappings in the setting of complete S-metric spaces. In addition, we provide some consequences of the established results. Also some examples are provided to validate the established results. Further, discuss the existence of solution to the system of nonlinear integral equations. The results presented in this paper extend, generalize and unify several results in the existing literature (see, e.g., [1], [10], [23], [25] and others).

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