

ALMOST (i, j) - m - I -CONTINUOUS MULTIFUNCTIONS ON IDEAL BITOPOLOGICAL SPACES

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ABSTRACT. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $(i, j)mIO(X)$ be a minimal structure on X constructed by topologies τ_1, τ_2 and ideal I . We introduce almost (i, j) - m - I -continuous multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$ and obtain their characterizations by using $(i, j)mIO(X)$.

1. INTRODUCTION

Recently, (i, j) - I -open sets, (i, j) -semi- I -open sets, (i, j) -pre- I -open sets, (i, j) - α - I -open sets, (i, j) - β - I -open sets, and (i, j) - b - I -open sets in an ideal bitopological space have been introduced and investigated.

The notions of minimal structures, m -spaces and m -continuity and M -continuity are introduced and investigated by the present authors in [17], [14]. Quite recently, as a unified form of the above open sets, the present authors [16] introduced the notion of a minimal structure $(i, j)mIO(X)$ determined by τ_1, τ_2 and I in an ideal bitopological space (X, τ_1, τ_2, I) and defined an (i, j) - m - I -continuous multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$. On the other hand, the authors [15] defined an almost m - I -continuous multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma)$ on an ideal topological space.

In this paper, as a generalization of the above two kinds of multifunctions, we introduce almost (i, j) - m - I -continuous multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$ and obtain their characterizations by using the minimal structure $(i, j)mIO(X)$.

2. PRELIMINARIES

Definition 2.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X . A subfamily m of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m -structure*) on X [17], [14] if $\emptyset, X \in m$.

By (X, m) we denote a nonempty set X with an m -structure m on X and call it an m -space. Each member of m is said to be *m -open* and the complement of an m -open set is said to be *m -closed*. The family of all m -open sets containing a point x is denoted by $m(x)$.

Definition 2.2. Let (X, m) be an m -space and A a subset of X . The *m -closure* of A and the *m -interior* of A are defined in [11] as follows:

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- (1) $mCl(A) = \cap\{F : A \subset F, X \setminus F \in m\}$,
- (2) $mInt(A) = \cup\{U : U \subset A, U \in m\}$.

Lemma 2.1. [11] *Let (X, m) be an m -space. For subsets A and B of X , the following properties hold:*

- (1) $mCl(X \setminus A) = X \setminus mInt(A)$ and $mInt(X \setminus A) = X \setminus mCl(A)$,
- (2) if $(X \setminus A) \in m$, then $mCl(A) = A$ and if $A \in m$, then $mInt(A) = A$,
- (3) $mCl(\emptyset) = \emptyset, mCl(X) = X, mInt(\emptyset) = \emptyset$ and $mInt(X) = X$,
- (4) if $A \subset B$, then $mCl(A) \subset mCl(B)$ and $mInt(A) \subset mInt(B)$,
- (5) $A \subset mCl(A)$ and $mInt(A) \subset A$,
- (6) $mCl(mCl(A)) = mCl(A)$ and $mInt(mInt(A)) = mInt(A)$.

Lemma 2.2. [17] *Let (X, m) be an m -space and A a subset of X . Then $x \in mCl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m(x)$.*

Definition 2.3. An m -structure m on a nonempty set X is said to have *property \mathcal{B}* [11] if the union of any family of subsets belonging to m belongs to m .

Lemma 2.3. [18] *Let (X, m) be an m -space and m have property \mathcal{B} . Then for a subset A of X , the following properties hold:*

- (1) $A \in m$ if and only if $mInt(A) = A$,
- (2) A is m -closed if and only if $mCl(A) = A$,
- (3) $mInt(A) \in m$ and $mCl(A)$ is m -closed.

Throughout the present paper, (X, τ) and (Y, σ) (briefly X and Y) always denote topological spaces and (X, τ_1, τ_2) is a bitopological space. For a multifunction $F : X \rightarrow Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y . For any open set V of Y , we denote $V^+ = \{B \in \mathcal{P}(Y) : B \subset V\}$ and $V^- = \{B \in \mathcal{P}(Y) : B \cap V \neq \emptyset\}$.

3. IDEAL BITOPOLOGICAL SPACES

The notion of ideals has been introduced in [8] and [21] and further investigated in [7].

A nonempty collection I of subsets of a nonempty set X is called an *ideal* on X [8], [21] if it satisfies the following two conditions:

- (1) $A \in I$ and $B \subset A$ implies $B \in I$,
- (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I) . Let (X, τ, I) be an ideal topological space. For any subset A of X , $A^*(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [7]. Hereafter $A^*(I, \tau)$ is simply denoted by A^* . It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator on X and the topology generated by Cl^* is denoted by τ^* .

Lemma 3.1. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X . Then the following properties hold:*

- (1) $A \subset B$ implies $\text{Cl}^*(A) \subset \text{Cl}^*(B)$,
- (2) $\text{Cl}^*(X) = X$ and $\text{Cl}^*(\emptyset) = \emptyset$,
- (3) $\text{Cl}^*(A) \cup \text{Cl}^*(B) = \text{Cl}^*(A \cup B)$.

Proof. This follows from $\text{Cl}^*(A) = A^* \cup A$ and Theorem 2.3 (a), (e) of [7]. \square

Definition 3.1. Let (X, τ_1, τ_2, I) be an ideal bitopological space. For any subset A of X , $A_j^*(I, \tau_j) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau_j(x)\}$, where $\tau_j(x) = \{U \in \tau_j : x \in U\}$. We put $j\text{Cl}^*(A) = A \cup A_j^*$ for every subset A of X . $A_j^*(I, \tau_j)$ is simply denoted by A_j^* .

Definition 3.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space. A subset A of X is said to be

- (1) (i, j) - I -open [5] if $A \subset i\text{Int}(A_j^*)$, for $i \neq j, i, j = 1, 2$,
- (2) (i, j) - α - I -open [6] if $A \subset i\text{Int}(j\text{Cl}^*(i\text{Int}(A)))$, for $i \neq j, i, j = 1, 2$,
- (3) (i, j) -semi- I -open [4] if $A \subset j\text{Cl}^*(i\text{Int}(A))$, for $i \neq j, i, j = 1, 2$,
- (4) (i, j) -pre- I -open [2] if $A \subset i\text{Int}(j\text{Cl}^*(A))$, for $i \neq j, i, j = 1, 2$,
- (5) (i, j) - b - I -open [19] if $A \subset i\text{Int}(j\text{Cl}^*(A)) \cup j\text{Cl}^*(i\text{Int}(A))$, for $i \neq j, i, j = 1, 2$,
- (6) (i, j) - β - I -open [3] if $A \subset j\text{Cl}(i\text{Int}(j\text{Cl}^*(A)))$, for $i \neq j, i, j = 1, 2$,
- (7) (i, j) -weakly semi- I -open if $A \subset j\text{Cl}^*(i\text{Int}(j\text{Cl}(A)))$, for $i \neq j, i, j = 1, 2$,
- (8) (i, j) -weakly b - I -open [20] if $A \subset j\text{Cl}(i\text{Int}(j\text{Cl}^*(A))) \cup j\text{Cl}^*(i\text{Int}(j\text{Cl}(A)))$, for $i \neq j, i, j = 1, 2$,
- (9) (i, j) -strongly β - I -open if $A \subset j\text{Cl}^*(i\text{Int}(j\text{Cl}^*(A)))$, for $i \neq j, i, j = 1, 2$.

The family of all (i, j) - I -open (resp. (i, j) - α - I -open, (i, j) -semi- I -open, (i, j) -pre- I -open, (i, j) - b - I -open, (i, j) - β - I -open, (i, j) -weakly semi- I -open, (i, j) -weakly b - I -open, (i, j) -strongly β - I -open) sets in an ideal bitopological space (X, τ_1, τ_2, I) is denoted by $(i, j)IO(X)$ (resp. $(i, j)\alpha IO(X)$, $(i, j)SIO(X)$, $(i, j)PIO(X)$, $(i, j)BIO(X)$, $(i, j)\beta IO(X)$, $(i, j)WSIO(X)$, $(i, j)WBIO(X)$, $(i, j)S\beta IO(X)$).

Definition 3.3. By $(i, j)mIO(X)$, we denote each one of the families $(i, j)IO(X)$, $(i, j)\alpha IO(X)$, $(i, j)SIO(X)$, $(i, j)PIO(X)$, $(i, j)BIO(X)$, $(i, j)\beta IO(X)$, $(i, j)WSIO(X)$, $(i, j)WBIO(X)$ and $(i, j)S\beta IO(X)$.

Lemma 3.2. *Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then $(i, j)mIO(X)$ is a minimal structure and has property \mathcal{B} .*

Proof. By Lemmas 2.1(3) and 3.1(2), $(i, j)mIO(X)$ is a minimal structure on X . It follows from Lemmas 2.1(4) and 3.1(1) that $(i, j)mIO(X)$ has property \mathcal{B} . \square

Definition 3.4. Let (X, τ_1, τ_2, I) be an ideal bitopological space. For a subset A of X , we define $(i, j)m\text{Cl}_I(A)$ and $(i, j)m\text{Int}_I(A)$ as follows:

- (1) $(i, j)m\text{Cl}_I(A) = \cap\{F : A \subset F, X \setminus F \in (i, j)mIO(X)\}$,
- (2) $(i, j)m\text{Int}_I(A) = \cup\{U : U \subset A, U \in (i, j)mIO(X)\}$.

Lemma 3.3. *Let (X, τ_1, τ_2, I) be an ideal bitopological space and A, B be subsets of X . Then the following properties hold:*

- (1) $A \subset (i, j)m\text{Cl}_I(A)$ and $(i, j)m\text{Cl}_I(A)$ is (i, j) - m - I -closed,
- (2) A is (i, j) - m - I -closed if and only if $A = (i, j)m\text{Cl}_I(A)$,

- (3) $(i, j)mCl_I(\emptyset) = \emptyset$ and $(i, j)mCl_I(X) = X$,
- (4) if $A \subset B$, then $(i, j)mCl_I(A) \subset (i, j)mCl_I(B)$,
- (5) $(i, j)mCl_I((i, j)mCl_I(A)) = (i, j)mCl_I(A)$,
- (6) $x \in (i, j)mCl_I(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i, j)mIO(X)$ such that $x \in U$.

Lemma 3.4. *Let (X, τ_1, τ_2, I) be an ideal bitopological space and A, B be subsets of X . Then the following properties hold:*

- (1) $(i, j)mInt_I(A) \subset A$ and $(i, j)mInt_I(A)$ is (i, j) - m - I -open,
- (2) A is (i, j) - m - I -open if and only if $A = (i, j)mInt_I(A)$,
- (3) $(i, j)mInt_I(\emptyset) = \emptyset$ and $(i, j)mInt_I(X) = X$,
- (4) if $A \subset B$, then $(i, j)mInt_I(A) \subset (i, j)mInt_I(B)$,
- (5) $(i, j)mInt_I((i, j)mInt_I(A)) = (i, j)mInt_I(A)$,
- (6) $x \in (i, j)mInt_I(A)$ if and only if there exists $U \in (i, j)mIO(X)$ such that $x \in U \subset A$.

Lemma 3.5. *Let (X, τ_1, τ_2, I) be an ideal bitopological space. For a subset A of X , the following properties hold:*

- (1) $(i, j)mCl_I(X \setminus A) = X \setminus (i, j)mInt_I(A)$,
- (2) $(i, j)mInt_I(X \setminus A) = X \setminus (i, j)mCl_I(A)$.

4. ALMOST (i, j) - m - I -CONTINUOUS MULTIFUNCTIONS

Let (X, τ) be a topological space and A a subset of X . The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $Cl(Int(A)) = A$ (resp. $Int(Cl(A)) = A$).

Definition 4.1. Let (X, τ) be a topological space. A subset A of X is said to be

- (1) α -open [12] if $A \subset Int(Cl(Int(A)))$,
- (2) semi-open [9] if $A \subset Cl(Int(A))$,
- (3) preopen [10] if $A \subset Int(Cl(A))$,
- (4) β -open [1] if $A \subset Cl(Int(Cl(A)))$.

The complement of a semi-open (resp. preopen, α -open, β -open) set is said to be *semi-closed* (resp. *preclosed*, α -closed, β -closed).

Definition 4.2. The intersection of all semi-closed (resp. preclosed, α -closed, β -closed) sets of X containing A is called the *semi-closure* (resp. *preclosure*, α -closure, β -closure) of A and is denoted by $sCl(A)$ (resp. $pCl(A)$, $\alpha Cl(A)$, $\beta Cl(A)$).

Definition 4.3. The union of all semi-open (resp. preopen, α -open, β -open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, α -interior, β -interior) of A and is denoted by $sInt(A)$ (resp. $pInt(A)$, $\alpha Int(A)$, $\beta Int(A)$).

Definition 4.4. A multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$ is said to be

- (1) almost (i, j) - m - I -continuous at a point $x \in X$ if for each open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in (i, j)mIO(X)$ containing x such that $F(U) \subset Int(Cl(V_1))$ and $F(U) \cap Int(Cl(V_2)) \neq \emptyset$ for every $u \in U$,
- (2) F is almost (i, j) - m - I -continuous if it has the property at each point $x \in X$.

Theorem 4.1. *For a multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

- (1) F is almost (i, j) - m - I -continuous at $x \in X$;
- (2) for any regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists a $U \in (i, j)mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$;
- (3) $x \in (i, j)m\text{Int}_I(F^+(G_1) \cap F^-(G_2))$ for every regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$;
- (4) $x \in (i, j)m\text{Int}_I(F^+(\text{Int}(\text{Cl}(V_1))) \cap F^-(\text{Int}(\text{Cl}(V_2))))$ for every open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$.

Proof. (1) \Rightarrow (2): This is obvious.

(2) \Rightarrow (3): Let G_1, G_2 be any regular open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then, there exists $U \in (i, j)mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Then, we have $x \in U \subset F^+(G_1) \cap F^-(G_2)$. Since $U \in (i, j)mIO(X)$, we obtain $x \in (i, j)m\text{Int}_I(F^+(G_1) \cap F^-(G_2))$.

(3) \Rightarrow (4): Let V_1, V_2 be any open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, $\text{Int}(\text{Cl}(V_1))$ and $\text{Int}(\text{Cl}(V_2))$ are regular open sets and $F(x) \in (\text{Int}(\text{Cl}(V_1)))^+ \cap (\text{Int}(\text{Cl}(V_2)))^-$. Therefore, by (3) we obtain $x \in (i, j)m\text{Int}_I[F^+(\text{Int}(\text{Cl}(V_1))) \cap F^-(\text{Int}(\text{Cl}(V_2)))]$.

(4) \Rightarrow (1): Let V_1, V_2 be any open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, by (4) $x \in (i, j)m\text{Int}_I[F^+(\text{Int}(\text{Cl}(V_1))) \cap F^-(\text{Int}(\text{Cl}(V_2)))]$. Therefore, there exists $U \in (i, j)mIO(X)$ containing x such that $U \subset F^+(\text{Int}(\text{Cl}(V_1))) \cap F^-(\text{Int}(\text{Cl}(V_2)))$. This implies that $F(U) \subset \text{Int}(\text{Cl}(V_1))$ and $F(u) \cap \text{Int}(\text{Cl}(V_2)) \neq \emptyset$ for each $u \in U$. Therefore, F is almost (i, j) - m - I -continuous at $x \in X$. \square

Lemma 4.2. [13] *Let A be any subset of a topological space (X, τ) . Then A is preopen if and only if $\text{Int}(\text{Cl}(A)) = \text{sCl}(A)$.*

Theorem 4.3. *For a multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

- (1) F is almost (i, j) - m - I -continuous;
- (2) for each $x \in X$ and any open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in (i, j)mIO(X)$ containing x such that $F(U) \subset \text{Int}(\text{Cl}(V_1))$ and $F(u) \cap \text{Int}(\text{Cl}(V_2)) \neq \emptyset$ for every $u \in U$;
- (3) for each $x \in X$ and any regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in (i, j)mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$;
- (4) $F^+(G_1) \cap F^-(G_2) = (i, j)m\text{Int}_I(F^+(G_1) \cap F^-(G_2))$ for every regular open sets G_1, G_2 of Y ;
- (5) $F^-(K_1) \cup F^+(K_2) = (i, j)m\text{Cl}_I(F^-(K_1) \cup F^+(K_2))$ for every regular closed sets K_1, K_2 of Y ;
- (6) $F^+(V_1) \cap F^-(V_2) \subset (i, j)m\text{Int}_I(F^+(\text{sCl}(V_1)) \cap F^-(\text{sCl}(V_2)))$ for every open sets V_1, V_2 of Y ;
- (7) $(i, j)m\text{Cl}_I(F^-(\text{sInt}(K_1)) \cup F^+(\text{sInt}(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ for every closed sets K_1 and K_2 of Y ;
- (8) $(i, j)m\text{Cl}_I(F^-(\text{Cl}(\text{Int}(K_1))) \cup F^+(\text{Cl}(\text{Int}(K_2)))) \subset F^-(K_1) \cup F^+(K_2)$ for every closed sets K_1 and K_2 of Y ;

(9) $(i, j)mCl_I(F^-(Cl(Int(Cl(B_1)))) \cup F^+(Cl(Int(Cl(B_2)))))) \subset F^-(Cl(B_1)) \cup F^+(Cl(B_2))$
for every subsets B_1 and B_2 of Y .

Proof. (1) \Rightarrow (2): The proof is obvious.

(2) \Rightarrow (3): The proof is obvious.

(3) \Rightarrow (4): Let G_1, G_2 be any regular open sets of Y and $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \in G_1^+ \cap G_2^-$ and by (3) there exists $U \in (i, j)mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^+(G_1) \cap F^-(G_2)$. Hence $F^+(G_1) \cap F^-(G_2) \subset (i, j)mInt_I(F^+(G_1) \cap F^-(G_2))$. By Lemma 3.4, we have $F^+(G_1) \cap F^-(G_2) = (i, j)mInt_I(F^+(G_1) \cap F^-(G_2))$.

(4) \Rightarrow (1): Let V_1, V_2 be any open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, $Int(Cl(V_1))$ and $Int(Cl(V_2))$ are regular open sets of Y and $F(x) \in (Int(Cl(V_1)))^+ \cap (Int(Cl(V_2)))^-$.

Then, by (4) we have

$$x \in F^+(Int(Cl(V_1))) \cap F^-(Int(Cl(V_2))) = (i, j)mInt_I[F^+(Int(Cl(V_1))) \cap F^-(Int(Cl(V_2)))].$$

There exists $U \in (i, j)mIO(X)$ containing x such that

$$F(U) \subset Int(Cl(V_1)) \text{ and } F(u) \cap Int(Cl(V_2)) \neq \emptyset \text{ for each } u \in U.$$

Therefore, F is almost (i, j) - m - I -continuous.

(4) \Leftrightarrow (5): This follows from the fact that $F^+(Y \setminus B) = X \setminus F^-(B)$ and $F^-(Y \setminus B) = X \setminus F^+(B)$.

(4) \Rightarrow (6): Let V_1, V_2 be any open sets of Y . Then $Int(Cl(V_1))$ and $Int(Cl(V_2))$ are regular open sets of Y . Therefore, by Lemma 4.2 we have

$$\begin{aligned} F^+(V_1) \cap F^-(V_2) &\subset F^+(Int(Cl(V_1))) \cap F^-(Int(Cl(V_2))) \\ &= (i, j)mInt_I(F^+(Int(Cl(V_1))) \cap F^-(Int(Cl(V_2)))) \\ &= (i, j)mInt_I(F^+(sCl(V_1)) \cap F^-(sCl(V_2))). \end{aligned}$$

(6) \Rightarrow (7): Let K_1, K_2 be any closed sets of Y . Then $Y \setminus K_1, Y \setminus K_2$ are open sets and by (6)

$$\begin{aligned} X \setminus (F^-(K_1) \cup F^+(K_2)) &= F^+(Y \setminus K_1) \cap F^-(Y \setminus K_2) \\ &\subset (i, j)mInt_I(F^+(sCl(Y \setminus K_1)) \cap F^-(sCl(Y \setminus K_2))) \\ &= (i, j)mInt_I(F^+(Y \setminus sInt(K_1)) \cap F^-(Y \setminus sInt(K_2))) \\ &= (i, j)mInt_I(X \setminus F^-(sInt(K_1))) \cap (X \setminus F^+(sInt(K_2))) \\ &= X \setminus (i, j)mCl_I(F^-(sInt(K_1)) \cup F^+(sInt(K_2))). \end{aligned}$$

Therefore, we obtain $(i, j)mCl_I(F^-(sInt(K_1)) \cup F^+(sInt(K_2))) \subset F^-(K_1) \cup F^+(K_2)$.

(7) \Rightarrow (8): The proof is obvious since $sInt(K) = Cl(Int(K))$ for every closed set K .

(8) \Rightarrow (9): The proof is obvious.

(9) \Rightarrow (5): Let K_1, K_2 be any regular closed sets in Y . Then

$$\begin{aligned} (i, j)mCl_I(F^-(K_1) \cup F^+(K_2)) &= (i, j)mCl_I(F^-(Cl(Int(Cl(K_1)))) \cup F^+(Cl(Int(Cl(K_2)))))) \\ &\subset F^-(Cl(K_1)) \cup F^+(Cl(K_2)) = F^-(K_1) \cup F^+(K_2). \end{aligned}$$

Therefore, we obtain

$$(i, j)mCl_I[F^-(K_1) \cup F^+(K_2)] = F^-(K_1) \cup F^+(K_2). \quad \square$$

Corollary 4.4. For a multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) F is almost (i, j) - m - I -continuous;
- (2) $F^+(G_1) \cap F^-(G_2)$ is (i, j) - m - I -open in X for any regular open sets G_1, G_2 of

Y ;

(3) $F^-(K_1) \cup F^+(K_2)$ is (i, j) - m - I -closed in X for any regular closed sets K_1, K_2 of Y .

Proof. By Lemma 3.2, $(i, j)mIO(X)$ has property \mathcal{B} and the proofs are obvious by (4) and (5) of Theorem 4.3. \square

Theorem 4.5. For a multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) F is almost (i, j) - m - I -continuous;
- (2) $(i, j)mCl_I(F^-(G_1) \cup F^+(G_2)) \subset F^-(Cl(G_1)) \cup F^+(Cl(G_2))$ for every β -open sets G_1, G_2 of Y ;
- (3) $(i, j)mCl_I(F^-(G_1) \cup F^+(G_2)) \subset F^-(Cl(G_1)) \cup F^+(Cl(G_2))$ for every semi-open sets G_1, G_2 of Y ;
- (4) $F^+(V_1) \cap F^-(V_2) \subset (i, j)mInt_I(F^+(sCl(V_1)) \cap F^-(sCl(V_2)))$ for every preopen sets V_1, V_2 of Y .

Proof. (1) \Rightarrow (2): Let G_1, G_2 be any β -open sets of Y . Then $Cl(G_1)$ and $Cl(G_2)$ are regular closed sets. Since F is almost (i, j) - m - I -continuous, by Theorem 4.3 (5), we have $(i, j)mCl_I(F^-(G_1) \cup F^+(G_2)) \subset (i, j)mCl_I(F^-(Cl(G_1)) \cup F^+(Cl(G_2))) = F^-(Cl(G_1)) \cup F^+(Cl(G_2))$.

(2) \Rightarrow (3): This is obvious since every semi-open set is β -open.

(3) \Rightarrow (4): Let V_1, V_2 be any preopen sets of Y . Then, $Int(Cl(V_1))$ and $Int(Cl(V_2))$ are regular open sets. Hence, by (3) and Lemma 4.2 we have

$$\begin{aligned} & X \setminus (F^+(V_1) \cap F^-(V_2)) \\ &= (X \setminus F^+(V_1)) \cup (X \setminus F^-(V_2)) \\ &= F^-(Y \setminus V_1) \cup F^+(Y \setminus V_2) \\ &\supset F^-(Y \setminus Int(Cl(V_1))) \cup F^+(Y \setminus Int(Cl(V_2))) \\ &\supset (i, j)mCl_I[F^-(Y \setminus Int(Cl(V_1))) \cup F^+(Y \setminus Int(Cl(V_2)))] \\ &= (i, j)mCl_I[X \setminus F^+(Int(Cl(V_1))) \cup (X \setminus F^-(Int(Cl(V_2)))] \\ &= (i, j)mCl_I(X \setminus [F^+(Int(Cl(V_1))) \cap F^-(Int(Cl(V_2)))] \\ &= X \setminus (i, j)mInt_I[F^+(Int(Cl(V_1))) \cap F^-(Int(Cl(V_2)))] \\ &= X \setminus (i, j)mInt_I[F^+(sCl(V_1)) \cap F^-(sCl(V_2))]. \end{aligned}$$

Therefore, we obtain

$$F^+(V_1) \cap F^-(V_2) \subset (i, j)mInt_I(F^+(sCl(V_1)) \cap F^-(sCl(V_2))).$$

(4) \Rightarrow (1): Let V_1, V_2 be regular open sets of Y . Then V_1, V_2 are preopen sets and hence

$$\begin{aligned} & F^+(V_1) \cap F^-(V_2) \subset (i, j)mInt_I[F^+(sCl(V_1)) \cap F^-(sCl(V_2))] \\ &= (i, j)mInt_I[F^+(Int(Cl(V_1))) \cap F^-(Int(Cl(V_2)))] = (i, j)mInt_I[F^+(V_1) \cap F^-(V_2)]. \end{aligned}$$

Hence, $F^+(V_1) \cap F^-(V_2) = (i, j)mInt_I[F^+(V_1) \cap F^-(V_2)]$. By Theorem 4.3, F is almost (i, j) - m - I -continuous. \square

Lemma 4.6. [15] Let (Y, σ) be a topological space. Then the following properties hold:

- (1) $\alpha Cl(V) = Cl(V)$ for every β -open set V of Y ,
- (2) $pCl(V) = Cl(V)$ for every semi-open set V of Y .

Corollary 4.7. For a multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$, the following properties are equivalent:

- (1) F is almost (i, j) - m - I -continuous;
(2) $(i, j)mCl_I(F^-(G_1) \cup F^+(G_2)) \subset F^-(\alpha Cl(G_1)) \cup F^+(\alpha Cl(G_2))$ for every β -open sets G_1, G_2 of Y ;
(3) $(i, j)mCl_I(F^-(G_1) \cup F^+(G_2)) \subset F^-(pCl(G_1)) \cup F^+(pCl(G_2))$ for every semi-open sets G_1, G_2 of Y .

Proof. The proof follows from Lemma 4.6 and Theorem 4.5. \square

A subset A is said to be δ -open [22] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a δ -cluster point of A if $\text{Int}(\text{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x . The set of all δ -cluster points of A is called the δ -closure of A and is denoted by $\text{Cl}_\delta(A)$. The set $\{x \in X : x \in U \subset A \text{ for some regular open set } U \text{ of } X\}$ is called the δ -interior of A and is denoted by $\text{Int}_\delta(A)$.

Theorem 4.8. *For a multifunction $F : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

- (1) F is almost (i, j) - m - I -continuous;
(2) $(i, j)mCl_I[F^-(\text{Cl}(\text{Int}(\text{Cl}_\delta(B_1)))) \cup F^+(\text{Cl}(\text{Int}(\text{Cl}_\delta(B_2))))] \subset F^-(\text{Cl}_\delta(B_1)) \cup F^+(\text{Cl}_\delta(B_2))$ for every subsets B_1, B_2 of Y ;
(3) $(i, j)mCl_I[F^-(\text{Cl}(\text{Int}(\text{Cl}(B_1)))) \cup F^+(\text{Cl}(\text{Int}(\text{Cl}(B_2))))] \subset F^-(\text{Cl}_\delta(B_1)) \cup F^+(\text{Cl}_\delta(B_2))$ for every subsets B_1, B_2 of Y .

Proof. (1) \Rightarrow (2): Let B_1, B_2 be any subsets of Y . By Lemma 2 of [22], $\text{Cl}_\delta(B_1)$ and $\text{Cl}_\delta(B_2)$ are closed sets. Then, by Theorem 4.3, we obtain

$$(i, j)mCl_I[F^-(\text{Cl}(\text{Int}(\text{Cl}_\delta(B_1)))) \cup F^+(\text{Cl}(\text{Int}(\text{Cl}_\delta(B_2))))] \subset F^-(\text{Cl}_\delta(B_1)) \cup F^+(\text{Cl}_\delta(B_2)).$$

(2) \Rightarrow (3): This is obvious since $\text{Cl}(B) \subset \text{Cl}_\delta(B)$ for every subset B of Y .

(3) \Rightarrow (1): Let K_1, K_2 be any regular closed sets of Y . Then by (3), we have

$$\begin{aligned} & (i, j)mCl_I[F^-(K_1) \cup F^+(K_2)] \\ &= (i, j)mCl_I[F^-(\text{Cl}(\text{Int}(K_1))) \cup F^+(\text{Cl}(\text{Int}(K_2)))] \\ &= (i, j)mCl_I[F^-(\text{Cl}(\text{Int}(\text{Cl}(K_1)))) \cup F^+(\text{Cl}(\text{Int}(\text{Cl}(K_2))))] \\ &\subset F^-(\text{Cl}_\delta(K_1)) \cup F^+(\text{Cl}_\delta(K_2)) \\ &= F^-(K_1) \cup F^+(K_2). \end{aligned}$$

Therefore, $(i, j)mCl_I[F^-(K_1) \cup F^+(K_2)] \subset F^-(K_1) \cup F^+(K_2)$. Hence, by Theorem 4.3 F is almost (i, j) - m - I -continuous. \square

REFERENCES

- [1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β -open sets and β -continuous mappings, Bull. Fac. Sci. Assiut Univ. **12** (1983), 77–90.
- [2] M. Caldas, S. Jafari and N. Rajesh, Preopen sets in ideal bitopological spaces, Bol. Soc. Paran. Mat. (3) **29**(2) (2011), 61–68.
- [3] M. Caldas, S. Jafari and N. Rajesh, Some fundamental properties of β -open sets in ideal bitopological spaces, Eur. J. Pure Appl. Math. **6**(2) (2013), 247–255.
- [4] M. Caldas, S. Jafari and N. Rajesh, Semiopen sets in ideal bitopological spaces, Ann. Univ. Sci. Budapest. Math. **62** (2019), 93–110.
- [5] M. Caldas, S. Jafari, N. Rajesh and F. Smarandache, On \mathcal{I} -open sets and \mathcal{I} -continuous functions in ideal bitopological spaces, Selected papers of the 2018 International Conferences on Topology and its Applications (7-11), Dimitris Georgiou, Stavros Iliadis, Athanasios Megaritis, Ioannis Kougias (Guest Editors of the special volume) (2020), 44–56.

- [6] A. I. El-Maghrabi, M. Caldas, S. Jafari, R. M. Latif, A. Nasef, N. Rajesh and S. Shanthi, *Properties of ideal bitopological α -open sets*, Sci. Stud. Res. Ser. Math. Inform. **27**(2) (2017), 15–36.
- [7] D. Janković and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly **97** (1990), 295–310.
- [8] K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
- [9] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.
- [10] S. A. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt. **53** (1982), 47–53.
- [11] H. Maki, K. C. Rao and A. Nagoor Gani, *On generalizing semi-open and preopen sets*, Pure Appl. Math. Sci. **49** (1999), 17–29.
- [12] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.
- [13] T. Noiri, *Almost quasi-continuous functions*, Bull. Inst. Math. Acad. Sinica **18** (1990), 321–332.
- [14] T. Noiri and V. Popa, *On upper and lower M -continuous multifunctions*, Filomat **14** (2000), 73–86.
- [15] T. Noiri and V. Popa, *Almost m - I -continuous multifunctions*, Questions Answers General Topology **42** (2024), 113–122.
- [16] T. Noiri and V. Popa, *On (i, j) - m - I -continuous multifunctions in ideal bitopological spaces*, Ann. Univ. Oradea, Ser. Math. (to appear).
- [17] V. Popa and T. Noiri, *On M -continuous functions*, Anal. Univ. "Dunarea de Jos" Galați, Ser. Mat. Fiz. Mec. Teor. (2) **18**(23) (2000), 31–41.
- [18] V. Popa and T. Noiri, *A unified theory of weak continuity for functions*, Rend. Circ. Mat. Palermo (2) **51** (2002), 439–464.
- [19] D. J. Sarma, *bI -open sets in ideal bitopological spaces*, Int. J. Pure Appl. Math. **105**(1) (2015), 7–18.
- [20] D. J. Sarma, A. Kilicman and L. N. Mishra, *A new type of weak open sets via idealization in bitopological spaces*, Malays. J. Math. Sci. **15**(2) (2021), 189–197.
- [21] R. Vaidyanathaswami, *The localization theory in set-topology*, Proc. Indian Acad. Sci. **20** (1945), 51–62.
- [22] N. V. Velicko, *H -closed topological spaces*, Amer. Math. Soc. Transl. (2) **78** (1968), 103–118.

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