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WELL-POSEDNESS AND INDIRECT STABILIZATION OF THE NON-DEGENERATE SCHRÖDINGER EQUATION

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Abstract. The aim of this paper is to study the boundary stabilization of the non-degenerate Schrödinger equation by a non-local operator of the generalized Riemann-Liouville integral type. In this study, the semigroup theory is applied to treat the existence and uniqueness part at first. Further, by means of the Borichev-Tomilov [12] theorem, the polynomial stability will be achieved after the failure of the exponential stability that is observed for this kind of dissipation laws.

1. Introduction

In this article, we address the Schrödinger equation as it is adapted to a fractional integral boundary dissipation. The system goes as follows:

$$\begin{cases} \Phi_t(x,t) + i(a(x)\Phi_x)_x(x,t) = 0, & (0,L)\times(0,+\infty), \\ \Phi(0,t) = 0, & t \geq 0, \\ (a\Phi_x)(L,t) = -i\sigma \mathfrak{I}^{1-\alpha,\eta}\Phi(L,t), & t \geq 0, \\ \Phi(x,0) = \Phi_0(x), & x \in (0,L), \end{cases} \tag{1.1}$$

where $\sigma > 0$, $\eta \geq 0$ and $a(x)$ is a function that will be specified later. The notation $\mathfrak{I}^{1-\alpha,\eta}$ stands for the generalized Riemann-Liouville's fractional Integral of order $0 < 1-\alpha < 1$, with respect to the time variable t (see Choi and MacCamy [20] and E. Blanc, G. Chiavassa, and B. Lombard [11]). It is denoted as follows :

$$\mathfrak{I}^{1-\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} f(\tau) d\tau. \tag{1.2}$$

1.1. **Historical Remainders.** The internal stabilization of the Schrödinger equation has drawn a lot of interest in recent years, using various types of damping kinds and domains, making it more comprehensive than boundary stabilization. Further information can be gained from [42, 38, 13, 18, 25, 26, 39, 40, 35] and the references therein.

Also, a lot of attention has been given to the boundary stabilization of this interesting equation, and one can start with [34] I. Lasiecka and R. Triggiani studied the existence, uniqueness, and uniform stability at the energy level in $L^2(\Omega)$ of the

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n-dimensional linear Schrödinger equation in a bounded open domain. The system is defined by:

$$\begin{cases} iu_t + \Delta u = 0, & \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = iu, & x \in \Gamma_1, t \geq 0, \\ u = 0, & x \in \Gamma_2, t \geq 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

The authors used semigroup theory to establish the well-posedness of the system and then derived, by the multiplier method, an optimal decay result. This same equation was studied in [39] replacing iu by $im(x)u_t$, and the authors established an exponential decay in L^2-norm and H^1-norm based on the same method as before imposing geometric control conditions on the boundary. In [45] the author considered this model in an almost star-shaped domain, and instead of $im(x)u_t$, he chose $a(x)u + l(x)u_t$, where $a(x)$ and $l(x)$ are two non-negative functions of class C^1, It was proved there that the energy associated decays exponentially and highlights the effect of the domain and the conditions chosen compared with the above cited papers. The fourth model was also discussed in [7]. The 1-dimensional case of this model was extensively studied with different kinds of controllers, as one can see [32, 31, 47, 21, 27, 48, 22, 22, 29, 16, 17, 15, 24, 23, 38, 1, 19, 43] and the references therein.

Additionally, in [42], S. Nicaise and S. Rebiai explored the impact of time delays in boundary or internal feedback stabilization of the multidimensional Schrödinger equation. By studying how time delays affect the stabilization process, they likely aimed to gain insights into the dynamics and control of such systems in various settings. The system is defined by:

$$\begin{cases} y_t(x,t) - i\Delta y(x,t) = 0, & x \in \Omega, t > 0, \\ y(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial y}{\partial \nu} = i\mu_1 y(x,t) + i\mu_2 y(x, t-\tau), & \in \Gamma_1, t > 0, \\ y(x, t-\tau) = f_0(x, t-\tau), & x \in \Omega, 0 < t < \tau, \\ y(x, 0) = y_0(x,t), & x \in \Omega, t > 0. \end{cases}$$

Where $\frac{\partial y}{\partial \nu}$ represents the normal derivative, τ denotes the time delay, and μ_1, μ_2 are positive real numbers. In both cases, under appropriate assumptions, they established sufficient conditions on the delay term to ensure the exponential stability of the solution. These findings were derived using suitable energy functional and certain observability estimates.

In [37] B. Mbodje investigated the asymptotic behavior of solutions to the wave equation with a boundary viscoelastic damper of the fractional derivative type; he used an augmented system technique to relax the difficulty caused by the fractional derivative. He demonstrated that the system is well-posed within the context of semigroup theory. Additionally, he proved that the associated semigroup is not exponentially stable, and from this time on, a lot of attention was given to this kind of damper, studying its effect as both internal and external damping, see [2, 3, 4, 5, 6, 9, 10, 14, 28, 46] and the references therein.

In [38] Meradjah et al, studied the Schrödinger equation with internal fractional Integral, of the generalized Riemann-Liouville [11, 20] integral type, damping. The system considered is as follows:

$$\begin{cases} iy_t(x,t) + a\Delta y(x,t) + i\gamma\partial^{\alpha,\eta} y(x,t) = 0, & \text{in } \Omega \times [0,\infty), \\ y(x,t) = 0, & \text{on } \Omega \times [0,\infty), \\ y(x,0) = y_0(x), & \text{on } \Omega. \end{cases}$$

The semigroup theory is used to prove well-posedness of this system, the produce a nice proof for the failure of the uniform stability and finally provide an optimal polynomial energy decay as $t^{-2/(1-\alpha)}$.

The aforesaid literature motivates and legitimizes, as a matter of fact, our study, and the paper goes as follows: Section 2, is devoted to some definitions and notations used in the study and we transform the system (1.1) into an augmented system in order to make the study easier. Section 3 deals with global existence, and this is obtained using semigroup theory. Section 4 is dedicated to nonuniform stability, based on the Pruss [44] theorem, which is a common effect if this kind of friction operates alone. Section 5 focused on twofold: first, the strong stability that results from the Arent-Batty [8] and Lyubich-Vu [36] theorems, and second, the polynomial stability based on the Borichev-Tomilov [12] theorem. Last but not least, Section 6 highlights the novelties of the paper and gives some possible and futuristic works.

2. The Augmented Model

This section is concerned with the reformulation of the model (1.1) into an augmented system. Before that we give some definitions, hypotheses, and notations used throughout the paper. We begin with the function used in the system, where $a \in C^1[0,L]$ and $a(x) \geq a_0 > 0 \ \forall \ x \in [0,L]$, also we need $a \in W^{1,\infty}(0,L)$. We denote by $H^1_{a,L}$ the particular Sobolev space given by

$$H^1_{a,L}(0,L) := \left\{ u \in H^1(0,L) | u(0) = 0 \right\}.$$

endowed with its usual norm and inner-product.
We need also the following claim.

Theorem 2.1. [37] *Let μ be the function*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}, \quad 0 < \alpha < 1. \tag{2.1}$$

The relationship between the system's input "$\mathcal{U}$" and output "$\mathcal{O}$" is then established.

$$\begin{aligned} \psi_t + (\xi^2 + \eta)\psi - \mathcal{U}(t)\mu(\xi) &= 0, \ \eta \geq 0, \ \mathbb{R} \times (0, +\infty), & (2.2) \\ \psi(\xi, 0) &= 0, & (2.3) \\ \mathcal{O}(t) &= \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi)\psi d\xi. & (2.4) \end{aligned}$$

Is given by :

$$\mathcal{O}(t) = \mathcal{I}^{1-\alpha,\eta}\mathcal{U}(t).$$

Where

$$\mathcal{I}^{\alpha,\eta} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau) d\tau.$$

Lemma 2.2. [10] *If* $\varrho \in D_\eta = \mathbb{C}\backslash]-\infty, -\eta[$ *then :*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\varrho + \xi^2 + \eta} d\xi = c(\varrho + \eta)^{\alpha - 1}. \tag{2.5}$$

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{|\varrho + \xi^2 + \eta|^2} d\xi = c(\varrho + \eta)^{\alpha - 2}. \tag{2.6}$$

Based on Theorem 2.1, system (1.1) can be transformed in an equivalent way to

$$\begin{cases} \Phi_t + i(a\Phi_x)_x = 0, & (0, L) \times (0, +\infty), \\ \psi_t + (\xi^2 + \eta)\psi - \mu(\xi)\Phi(L, t) = 0, & \mathbb{R} \times (0, +\infty), \\ \Phi(0, t) = 0, & t \geq 0, \\ (a\Phi_x)(L, t) = -i\widetilde{\kappa} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\psi d\xi, & t \geq 0, \\ \Phi(x, 0) = \Phi_0(x), & x \in (0, L), \\ \psi(\xi, 0) = \psi_0(\xi), & \xi \in \mathbb{R}. \end{cases} \tag{2.7}$$

Where $\widetilde{\kappa} = \sigma\pi^{-1}\sin(\alpha\pi)$. We define the energy of the solution as follows:

$$\mathcal{E}(t) = \frac{1}{2}\left[||\Phi||^2_{L^2(0,L)} + \widetilde{\kappa}||\psi||^2_{L^2(\mathbb{R})} \right]. \tag{2.8}$$

$$\mathcal{E}'(t) = -\widetilde{\kappa} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi. \tag{2.9}$$

3. Global existence

In this section, we employ semigroup theory to establish the existence and uniqueness of solutions to system (2.7). We start by the energy space :

$$\mathcal{H} = L^2(0, L) \times L^2(\mathbb{R}). \tag{3.1}$$

We define the inner product of $\mathcal{Z}, \widetilde{\mathcal{Z}} \in \mathcal{H}$, where $\widetilde{\mathcal{Z}} \in \mathcal{H}$ with $\mathcal{Z} = (\Phi, \psi)^T$, $\widetilde{\mathcal{Z}} = (\widetilde{\Phi}, \widetilde{\psi})^T$, as follows:

$$< \mathcal{Z}, \widetilde{\mathcal{Z}} >_{\mathcal{H}} = \int_0^L \Phi\overline{\widetilde{\Phi}} dx + \widetilde{\kappa} \int_{-\infty}^{+\infty} \psi\overline{\widetilde{\psi}} d\xi. \tag{3.2}$$

To transform problem (2.7) into an abstract problem within the Hilbert space $\mathcal{H}$, let $\mathcal{K} = (\Phi, \psi)^T$, and system (2.7) can be expressed as follows:

$$\begin{cases} \partial_t \mathcal{K}(t) = \mathcal{A}\mathcal{K}(t), \\ \mathcal{K}(0) = \mathcal{K}_0. \end{cases} \tag{3.3}$$

Where $\mathcal{K}_0 = (\Phi_0, \psi_0)$ and $\mathcal{A}$ is the operator :

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}. \tag{3.4}$$

Defined by :

$$\mathcal{A}\begin{bmatrix} \Phi \\ \psi \end{bmatrix} = \begin{bmatrix} -i(a\Phi_x)_x \\ -(\xi^2 + \eta)\psi + \mu(\xi)\Phi(L) \end{bmatrix}. \tag{3.5}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ (\Phi, \psi)^T \in \mathcal{H} \;\middle|\; \begin{array}{l} \Phi \in H^2(0,L) \cap H^1_{a,L}; \\ (a\Phi_x)(L) = -i\widetilde{\kappa} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\psi d\xi; \\ |\xi|\psi \in L^2(\mathbb{R}) \end{array} \right\}. \tag{3.6}$$

Using the aforementioned notations, equation (2.8) can be expressed simply as

$$\mathcal{E}(t) = \frac{1}{2}\|\mathcal{Z}\|^2_{\mathcal{H}}. \tag{3.7}$$

We are now in a position to assert and illustrate the following lemma.

Lemma 3.1. *Let (Φ, ψ) be a regular solution of system (2.7).Then*

$$\mathcal{E}'(t) = \frac{d}{dt}\frac{1}{2}\|\mathcal{Z}\|^2_{\mathcal{H}} = -\widetilde{\kappa}\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi. \tag{3.8}$$

Proof. Multiplying equation $(2.7)_1$ by $\overline{\Phi}$ and integrating over (0,L) using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\Phi\|^2_{L^2(0,L)} - i\|\sqrt{a}\Phi_x\|^2_{L^2(0,L)} + \widetilde{\kappa}\int_{-\infty}^{+\infty} \mu(\xi)\overline{\Phi(1,t)}\psi d\xi = 0. \tag{3.9}$$

Multiplying equation $(2.7)_2$ by $\widetilde{\kappa}\overline{\psi}$ and integrating over $\mathbb{R}$, we get

$$\frac{\widetilde{\kappa}}{2}\frac{d}{dt}\|\psi\|^2_{L^2(\mathbb{R})} + \widetilde{\kappa}\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi - \widetilde{\kappa}\int_{-\infty}^{+\infty} \mu(\xi)\,\Phi(1,t)\overline{\psi} d\xi = 0. \tag{3.10}$$

Summing equations (3.9) and (3.10), taking the real part, and integrating over (0,t), we get

$$\mathcal{E}(t) = \frac{1}{2}\left[\|\Phi\|^2_{L^2(0,L)} + \widetilde{\kappa}\|\psi\|^2_{L^2(\mathbb{R})}\right].$$

And

$$\mathcal{E}'(t) = -\widetilde{\kappa}\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi.$$

□

Before going further, we recall the following theorem that will be used to prove global existence.

Theorem 3.2. [30] *Let $\mathcal{B}$ be a densely defined and closed linear operator in a Banach space $\mathcal{X}$. If both $\mathcal{B}$ and $\mathcal{B}^*$ are dissipative, then $\mathcal{B}$ generates a C_0-semigroup of contractions on $\mathcal{X}$.*

The well-posedness of problem (2.7) is ensured by the following theorem.

Theorem 3.3. *(Existence and uniqueness)*

- *If $\mathcal{K}_0 \in \mathcal{H}$, then system (2.7) has a unique weak solution $\mathcal{K} \in C^0(\mathbb{R}^+; \mathcal{H})$.*
- *If $\mathcal{K}_0 \in \mathcal{D}(\mathcal{A})$, then system (2.7) has a unique strong solution*

$$\mathcal{K} \in C^0(\mathbb{R}^+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H}).$$

We must first prove the following theorem in order to establish the above theorem.

Lemma 3.4. *Let $\mathcal{A}$ be the operator defined by equation (3.5). Then its adjoint operator is defined by*

$$\mathcal{A}^* \begin{bmatrix} \Phi \\ \psi \end{bmatrix} = \begin{bmatrix} i(a\Phi_x)_x \\ -(\xi^2+\eta)\psi - \mu(\xi)\Phi(L) \end{bmatrix}. \tag{3.11}$$

The domain of $\mathcal{A}^*$ is given by

$$\mathcal{D}(\mathcal{A}^*) = \left\{ (\Phi,\psi)^T \in \mathcal{H} \;\middle|\; \begin{array}{l} \Phi \in H^2(0,L) \cap H^1_{a,L}; \\ (a\Phi_x)(L) = -i\widetilde{\kappa} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\psi d\xi; \\ |\xi|\psi \in L^2(\mathbb{R}) \end{array} \right\}. \tag{3.12}$$

Furthermore $\mathcal{A}$ is densely defined and closed and both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative.

Proof. Let $\mathcal{Z} \in \mathcal{D}(\mathcal{A})$ and $\widetilde{\mathcal{Z}} \in \mathcal{D}(\mathcal{A}^*)$ such that $\mathcal{Z} = (\Phi,\psi)^T$ and $\widetilde{\mathcal{Z}} = (\widetilde{\Phi},\widetilde{\psi})^T$.

$$< \mathcal{A}\mathcal{Z}, \widetilde{\mathcal{Z}} >_{\mathcal{H}} = -i\int_0^L (a\Phi_x)_x \overline{\widetilde{\Phi}} dx + \widetilde{\kappa}\int_{-\infty}^{+\infty} \left[-(\xi^2+\eta)\psi + \mu(\xi)\Phi(L) \right] \overline{\widetilde{\psi}} d\xi. \tag{3.13}$$

Performing a part-by-part integration in the first integral gives us

$$-i\int_0^L (a\Phi_x)_x \overline{\widetilde{\Phi}} dx = -\overline{\widetilde{\Phi}(L)}\widetilde{\kappa}\int_{-\infty}^{+\infty} \mu(\xi)\psi d\xi - \Phi(L)\widetilde{\kappa}\int_{-\infty}^{+\infty} \mu(\xi)\overline{\widetilde{\psi}} d\xi - i\int_0^L \Phi\overline{(a\widetilde{\Phi}_x)_x} dx. \tag{3.14}$$

Inserting equation(3.14) into equation (3.13) and simplifying gives us

$$\begin{aligned} < \mathcal{A}\mathcal{Z}, \widetilde{\mathcal{Z}} >_{\mathcal{H}} &= -i\int_0^L \Phi\overline{(a\widetilde{\Phi}_x)_x} dx + \widetilde{\kappa}\int_{-\infty}^{+\infty} [-(\xi^2+\eta)\overline{\widetilde{\psi}} - \mu(\xi)\overline{\widetilde{\Phi}(L)}]\psi d\xi \\ &=< \mathcal{Z}, \mathcal{A}^*\widetilde{\mathcal{Z}} >_{\mathcal{H}} . \end{aligned} \tag{3.15}$$

It is not hard to check that $\mathcal{A}$ densely defined and closed; also, it is dissipative since

$$\Re e < \mathcal{A}\mathcal{Z}, \mathcal{Z} >_{\mathcal{H}} = \mathcal{E}'(t) = -\widetilde{\kappa}\int_{-\infty}^{+\infty} (\xi^2+\eta)|\psi|^2 d\xi \leq 0. \tag{3.16}$$

Based on the above computation, we infer that $\mathcal{A}^*$ is similarly dissipative.

$$\Re e < \mathcal{A}^*\mathcal{Z}, \mathcal{Z} >_{\mathcal{H}} = -\widetilde{\kappa}\int_{-\infty}^{+\infty} (\xi^2+\eta)|\psi|^2 d\xi. \tag{3.17}$$

And the proof is complete. □

Proof. Theorem 3.3, This theorem follows as a consequence of Lemma 3.4, Theorem 3.2 and general theory of Semigroup. □

4. NON UNIFORM STABILITY

This section addresses the lack of exponential stability of system (2.7), and before delving into that, we present the following theorem.

Theorem 4.1. [44] *Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{S}$. Then $S(t)$ is analytic if and only if*

$$i\mathbb{R} \in \rho(A), \tag{4.1}$$

and

$$\overline{\lim_{|s|\longrightarrow\infty}} \|(isI - A)^{-1}\|_{\mathcal{L}(\mathcal{S})} < \infty. \tag{4.2}$$

We can now claim that:

Theorem 4.2. *The semi-group generated by the operator $\mathcal{A}$ is not exponentially stable.*

The proof of Theorem 4.2 follows from the following two lemmas.

Lemma 4.3. *For $\eta = 0$, $\varrho \in \mathbb{C}$.Then $\varrho = 0$ is not in the resolvent set of $\mathcal{A}$.*

Proof. If no , for $\mathcal{G} = (i(a(x)\cos(x))_x, 0)$ denote $\mathcal{Z} = (\Phi, \psi)$ as the vector associated with $\mathcal{G}$ and we write

$$\begin{cases} i(a(x)\Phi_x)_x = i(a(x)\cos(x))_x, \\ \xi^2\psi - \mu(\xi)\Phi(L) = 0. \end{cases} \tag{4.3}$$

Then $\psi = sin(L)|\xi|^{\frac{2\alpha-5}{2}}$, which is not an element of $L^2(\mathbb{R})$. □

Lemma 4.4. *There exist $N \in \mathbb{N}$, such that :*

$$\{\varrho_k\}_{k\in\mathbb{Z}^*, k\geq N} \subset \sigma(\mathcal{A}), \tag{4.4}$$

where

$$\begin{gathered} \varrho_k = i\frac{(2k+1)^2\pi^2}{4m^2} + \widetilde{\beta}\left(\cos(\tfrac{\pi}{2}(1-\alpha)) - i\sin\left(\tfrac{\pi}{2}(1-\alpha)\right)\right) + o\left(\frac{1}{k^{2-2\alpha}}\right), \\ \varrho_k = \overline{\varrho_k} \ \ if \ \ k \leq -N, \end{gathered} \tag{4.5}$$

and $\tilde{\beta} \leq 0$. Furthermore for each $| k |\geq N$ the eigenvalues ϱ_k are simple.

Proof. We show that the semi-group associated with $\mathcal{A}$ is not exponentially stable because it admits a sequence of eigenvalues that approach the imaginary axis. We shall use Rouché's theorem and asymptotic expansions to do this.
Step 1: Let $\varrho \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$, then

$$\begin{cases} \varrho\Phi + i(a\Phi_x)_x = 0, \\ \varrho\psi + (\xi^2 + \eta)\psi - \mu(\xi)\Phi(L) = 0. \end{cases} \tag{4.6}$$

Equation $(4.6)_2$, gives us

$$\psi = \frac{\mu(\xi)\Phi(L)}{\varrho + \xi^2 + \eta}. \tag{4.7}$$

Using the above equation, system(4.6) reduces to

$$\begin{cases} \Phi_{xx} + \dfrac{a_x(x)}{a(x)}\Phi_x - \dfrac{i\varrho}{a(x)}\Phi = 0, \\ \Phi(0) = 0, \\ (a\Phi_x)(L) + i\sigma(\varrho + \eta)^{\alpha-1}\Phi(L) = 0. \end{cases} \tag{4.8}$$

By a spatial-scale transformation in x given as follows

$$\omega(z) = \Phi(x), \quad z = \frac{L}{m}\int_0^x \frac{1}{\sqrt{a(s)}}ds, \quad z \in (0, L), \tag{4.9}$$

where $m = \displaystyle\int_0^L \frac{1}{\sqrt{a(s)}}ds$, system(4.8) will take the form

$$\begin{cases} \omega''(z) + \dfrac{ma_x(x)}{2L\sqrt{a(x)}}\omega'(z) - \dfrac{im^2\varrho}{L^2}\omega(z) = 0, \\ \omega(0) = 0, \\ \omega'(L) + \dfrac{im\sigma}{L\sqrt{a(L)}}(\varrho+\eta)^{\alpha-1}\omega(L) = 0. \end{cases} \tag{4.10}$$

We can get a more simplified system by introducing another invertible transformation (See [41]):

$$\theta(z) = e^{\frac{1}{2}\int_0^z \hat{a}(s)ds}\omega(z), \quad z \in (0, L), \tag{4.11}$$

where $\hat{a}(z) = \dfrac{m}{2L}\dfrac{a_x(x)}{\sqrt{a(x)}}$, and we get

$$\begin{cases} \theta''(z) - \left(\dfrac{1}{2}\hat{a}'(z) + \dfrac{1}{4}\hat{a}^2(z) + \dfrac{im^2\varrho}{L^2}\right)\theta(z) = 0, \\ \theta(0) = 0, \\ \theta'(L) + \left(-\dfrac{1}{2}\hat{a}(L) + \dfrac{im\sigma}{L\sqrt{a(L)}}(\varrho+\eta)^{\alpha-1}\right)\theta(L) = 0. \end{cases} \tag{4.12}$$

In order to asymptotically estimate the solutions to the above eigenvalue problem, we are going to borrow some ideas from [41]. One can get for equation $(4.12)_1$ two linearly independent asymptotic fundamental solutions as:

$$\begin{cases} \theta_1(z) = e^{\frac{\gamma_1 m\sqrt{\varrho}}{L}z}\left(1 + \dfrac{\widetilde{\theta}_1(z)}{im\sqrt{\varrho}} + o\left(\dfrac{1}{\varrho}\right)\right), \\ \theta_2(z) = e^{\frac{-\gamma_1 m\sqrt{\varrho}}{L}z}\left(1 + \dfrac{\widetilde{\theta}_2(z)}{im\sqrt{\varrho}} + o\left(\dfrac{1}{\varrho}\right)\right), \end{cases} \tag{4.13}$$

with the formulas of their derivatives

$$\begin{cases} \dfrac{d}{dz}\theta_1(z) = \frac{\gamma_1 m\sqrt{\varrho}}{L}e^{\frac{\gamma_1 m\sqrt{\varrho}}{L}z}\left(1 + \dfrac{\widetilde{\theta}_1(z)}{im\sqrt{\varrho}} + o\left(\dfrac{1}{\varrho}\right)\right), \\ \dfrac{d}{dz}\theta_2(z) = \frac{-\gamma_1 m\sqrt{\varrho}}{L}e^{\frac{-\gamma_1 m\sqrt{\varrho}}{L}z}\left(1 + \dfrac{\widetilde{\theta}_2(z)}{im\sqrt{\varrho}} + o\left(\dfrac{1}{\varrho}\right)\right), \end{cases} \tag{4.14}$$

where $\gamma_1 = e^{i\frac{\pi}{4}}$ and

$$\begin{cases} \widetilde{\theta}_1(z) = -\dfrac{i}{2}\displaystyle\int_0^z \left(\dfrac{1}{2}\hat{a}'(s) + \dfrac{1}{4}\hat{a}^2(s)\right)ds, \\ \widetilde{\theta}_1(z) = \dfrac{i}{2}\displaystyle\int_0^z \left(\dfrac{1}{2}\hat{a}'(s) + \dfrac{1}{4}\hat{a}^2(s)\right)ds. \end{cases}$$

Foe simplicity, we introduce the notation $[a]_n := a + O(\sqrt{\varrho}^{-n})$ for $n = 1, 2$, we get

$$\theta_1(z) = c_1\theta_1(z) + c_2\theta_2(z). \tag{4.15}$$

Using equations $(4.12)_2$ and $(4.12)_3$, we conclude that

$$M(\varrho)C = \begin{pmatrix} [1]_2 & [1]_2 \\ \left[\left(\sqrt{\varrho} + \frac{\gamma_1\sigma}{\sqrt{a(L)}}\varrho^{\alpha-1}\right)e^{\gamma_1 m\sqrt{\varrho}}\right]_0 & \left[\left(-\sqrt{\varrho} + \frac{\gamma_1\sigma}{\sqrt{a(L)}}\varrho^{\alpha-1}\right)e^{-\gamma_1 m\sqrt{\varrho}}\right]_0 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0. \tag{4.16}$$

Then equation (4.12) will have a non-trivial solution if and only if $f(\varrho) = det(M(\varrho)) = 0$, with

$$\begin{aligned} f(\varrho) &= \left(-\sqrt{\varrho} + \frac{\gamma_1\sigma}{\sqrt{a(L)}}\varrho^{\alpha-1}\right)e^{-\gamma_1 m\sqrt{\varrho}} - \left(\sqrt{\varrho} + \frac{\gamma_1\sigma}{\sqrt{a(L)}}\varrho^{\alpha-1}\right)e^{\gamma_1 m\sqrt{\varrho}} + O(1) \\ &= -\sqrt{\varrho}e^{-\gamma_1 m\sqrt{\varrho}}\left(e^{2\gamma_1 m\sqrt{\varrho}} + 1 + \frac{\gamma_1\sigma}{\sqrt{\varrho}^{3-2\alpha}\sqrt{a(L)}}\left(e^{2\gamma_1 m\sqrt{2\varrho}} - 1\right) + O(\sqrt{\varrho}^{-1})\right), \end{aligned}$$

we set

$$\widetilde{f}(\varrho) = f_0(\varrho) + \frac{f_1(\varrho)}{\varrho^{\frac{3-2\alpha}{2}}} + o\left(\frac{1}{\sqrt{\varrho}^{3-2\alpha}}\right), \tag{4.17}$$

where

$$\begin{cases} f_0(\varrho) = e^{2\gamma_1 m\sqrt{\varrho}} + 1, \\ f_1(\varrho) = \frac{\gamma_1\sigma}{\sqrt{a(L)}}\left(e^{2\gamma_1 m\sqrt{\varrho}} - 1\right). \end{cases} \tag{4.18}$$

Step 2: Note that f_0 and f_1 remain bounded in the strip $\alpha_0 \le \Re e(\varrho) \le 0$. It is easy to check that the roots of f_0 are given by :

$$\varrho_k^0 = i\mu_k, \ where \ \mu_k = \frac{(2k+1)^2\pi^2}{4m^2}, \ k \in \mathbb{Z}. \tag{4.19}$$

Using Rouché's theorem, we deduce that $f(\varrho)$ admits an infinite of simple roots in the strip $\alpha_0 \le \Re e(\varrho) \le 0$ denoted by ϱ_k , with $|k| \ge k_0$, for k_0 large enough, such that :

$$\varrho_k = i\mu_k + o(1) \ as \ k \longrightarrow +\infty. \tag{4.20}$$

In an equivalent way

$$\varrho_k = i\mu_k + \varepsilon_k \ where \ \varepsilon_k \xrightarrow[k\to+\infty]{} 0. \tag{4.21}$$

Step 3: From equations (4.19) and (4.21), we can write

$$\varrho_k = i\frac{(2k+1)^2\pi^2}{4m^2} + \varepsilon_k. \tag{4.22}$$

Using equation (4.22), we get

$$e^{2\gamma_1 m\sqrt{\varrho}} = -1 - \frac{2m^2\varepsilon_k}{(2k+1)\pi} + o\left(\frac{\varepsilon_k}{k^2}\right). \tag{4.23}$$

Substituting equation (4.23) into equation (4.17), using the fact that $\widetilde{f}(\varrho_k) = 0$, we get

$$\begin{aligned} \widetilde{f}(\varrho_k) &= -\frac{2m^2\varepsilon_k}{(2k+1)\pi} - \frac{2\gamma_1\sigma}{(\varrho_k)^{\frac{3-2\alpha}{2}}\sqrt{a(L)}} + o(\varepsilon_k^2) \\ &= -\frac{m^2\varepsilon_k}{k\pi} - \frac{2m^{3-2\alpha}\gamma_1\sigma}{(k\pi)^{3-2\alpha}i^{\frac{3-2\alpha}{2}}\sqrt{a(L)}} + o(\varepsilon_k) + o\left(\frac{1}{k^{3-2\alpha}}\right) = 0, \end{aligned}$$

hence

$$\begin{aligned}\varepsilon_k &= -\frac{m^{1-2\alpha}\sigma}{\sqrt{a(L)}k^{2-2\alpha}\pi^{2-2\alpha}i^{1-\alpha}}\\ &= \begin{cases} -\dfrac{m^{1-2\alpha}\sigma}{\sqrt{a(L)}k^{2-2\alpha}\pi^{2-2\alpha}}\left(\cos\left(\frac{\pi}{2}(1-\alpha)\right)-i\sin\left(\frac{\pi}{2}(1-\alpha)\right)\right)+o\left(\dfrac{1}{k^{2-2\alpha}}\right) & for \quad k\geq 0,\\ -\dfrac{m^{1-2\alpha}\sigma}{\sqrt{a(L)}k^{2-2\alpha}\pi^{2-2\alpha}}\left(\cos\left(\frac{\pi}{2}(1-\alpha)\right)-i\sin\left(\frac{\pi}{2}(1-\alpha)\right)\right)+o\left(\dfrac{1}{k^{2-2\alpha}}\right) & for \quad k\leq 0.\end{cases}\end{aligned} \tag{4.24}$$

Equation (4.24) implies that

$$|k|^{2-2\alpha}\mathfrak{Re}(\varrho_k)\approx -\frac{m^{1-2\alpha}\sigma}{\sqrt{a(L)}\pi^{2-2\alpha}}\cos\left(\frac{\pi}{2}(1-\alpha)\right). \tag{4.25}$$

Choose $\widetilde{\beta} = -\dfrac{m^{1-2\alpha}\sigma}{\sqrt{a(L)}k^{2-2\alpha}\pi^{2-2\alpha}}$.

A substitution of equation (4.24) in equation (4.22), will then give the desired equation mentioned in (4.5). This implies that the C_0-semigroup of contraction $e^{\mathcal{A}t}$ is not uniformly stable in the energy space $\mathcal{H}$. □

Proof. Theorem 4.2.
Lemma 4.3 states that condition (4.2) is not met when $\eta = 0$, hence demonstrating the theorem.
The semigroup cannot be exponential when $\eta \neq 0$, as the branch mentioned in Lemma 4.4 goes toward the imaginary axis. This suggests that the theorem is not uniformly stable in the energy space $\mathcal{H}$ for the C_0-semigroup of contractions $e^{\mathcal{O}t}$. □

5. Asymptotic Stability

In this section, we are going to prove that system 2.7 decays strongly to zero by means of Arent-Batty [8] and Lyubich-Vu [36] theorems, and since the exponential decay has been excluded in the previous section, we will prove that it decays polynomially using the well-known Borichev-Tomilov [12]. We begin first with the strong stability.

5.1. **Strong stability of the system.**

Theorem 5.1. *The C_0-semigroup $e^{\mathcal{A}t}$ is strongly stable in $\mathcal{H}$. i.e: $\forall \mathcal{K}_0 \in \mathcal{H}$, the solution of system(2.7) satisfies*

$$\lim_{t\longmapsto\infty}\|e^{\mathcal{A}t}\mathcal{K}_0\| = 0.$$

To prove this theorem, we need the following two lemmas

Lemma 5.2. *Let $\mathcal{A}$ be the operator defined by equation (3.5), then $\mathcal{A}$ does not have eigenvalues in $i\mathbb{R}$.*

Proof. Let us suppose the contrary and argue by contradiction.
Case1:$\varrho \neq 0$ and take $\mathcal{Z} \in \mathcal{D}(\mathcal{A})$. Then

$$\mathcal{A}\mathcal{Z} = i\varrho\mathcal{Z}. \tag{5.1}$$

Equation (5.1) is equivalent to

$$\begin{cases} i\varrho\Phi + i(a\Phi_x)_x = 0, \\ i\varrho\psi + (\xi^2+\eta)\psi - \mu(\xi)\Phi(L) = 0. \end{cases} \tag{5.2}$$

Taking the inner-product with $\mathcal{Z} \in \mathcal{H}$ in (5.1) and the considerations in (3.8), we reach that

$$0 = \Re e < \mathcal{A}\mathcal{Z}, \mathcal{Z} >_{\mathcal{H}} = \mathcal{E}'(t) = -\widetilde{\kappa} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\psi|^2 d\xi. \tag{5.3}$$

$$\psi = 0, \ \ in \ \mathbb{R}.$$

Equation $(5.2)_2$, multiplied by $\frac{\mu(\xi)\overline{\Phi(L)}}{i\varrho+\xi^2+\eta}$ and integrated over $\mathbb{R}$, will give

$$\Phi(L) = 0.$$

We have to show that $\Phi = 0 \ \ in \ \ (0, L)$, we have

$$\begin{cases} \varrho\Phi + (a\Phi_x)_x = 0, \\ \Phi_x(L) = \Phi(L) = \Phi(0) = 0. \end{cases} \tag{5.4}$$

The proof of this fact is obviously related to the sign of ϱ, and it goes as follows:
Case:$\varrho > 0$**.** We are going to use some ideas from [46]. Let us consider the following function

$$\phi(x) = \int_0^x e^{\int_s^x \left|\frac{a_x(v)}{a(v)}\right| dv} ds, \quad \forall \ x \in [0, L]. \tag{5.5}$$

One can easily check that ϕ has the following properties

$$\begin{cases} \phi(0) = 0, \quad \phi(x) > 0, \qquad \forall \ x \in [0, L]. \\ \phi_x \geq 1, \qquad a\left(\dfrac{\phi}{a}\right)_x \geq 1. \end{cases} \tag{5.6}$$

Multiplying equation $(5.4)_1$ by $\phi\overline{\Phi_x}$, we get

$$\varrho \int_0^L \phi\overline{\Phi_x}\Phi dx + \int_0^L \phi\overline{\Phi_x}(a\Phi_x)_x dx = 0. \tag{5.7}$$

Performing an integration by parts on the right term of equation (5.7) and using the properties of ϕ given in equation (5.6), we obtain

$$\frac{\varrho}{2} \int_0^L \phi \frac{d}{dx}|\Phi|^2 dx - \int_0^L \phi_x a|\Phi_x|^2 dx - \frac{1}{2} \int_0^L \phi_x a(x) \frac{d}{dx}|\Phi_x|^2 dx = 0. \tag{5.8}$$

Another integration by parts in equation (5.8), using equations $(5.4)_2$ and (5.6), will give

$$\varrho \int_0^L \phi_x |\Phi|^2 dx + \int_0^L \left(\phi_x a - \phi a_x\right) |\Phi_x|^2 dx = 0. \tag{5.9}$$

From equation (5.9) one can easily deduce that $\Phi = 0,$ in $(0, L)$.
Case:$\varrho < 0$. Without losing generality we can assume that $\varrho = -k^2$ for some real k. Multiplying equation $(5.4)_1$ by $\overline{\Phi}$, we get

$$-k^2 \|\Phi\|^2_{L^2(0,L)} + \int_0^L \overline{\Phi}(a\Phi_x)_x dx = 0. \tag{5.10}$$

Performing an integration by parts on the right term of equation (5.10), we obtain

$$-k^2 \|\Phi\|^2_{L^2(0,L)} - \int_0^L a|\Phi_x|^2 dx = 0. \tag{5.11}$$

Which gives us $\Phi = 0,\ in\ (0, L)$.
Case2: $\varrho = 0$.

$$\mathcal{A}\mathcal{Z} = 0. \tag{5.12}$$

Equation (5.12) is equivalent to

$$\left\{ \begin{array}{l} (a\Phi_x)_x = 0, \\ (\xi^2 + \eta)\psi - \mu(\xi)\Phi(L) = 0. \end{array} \right. \tag{5.13}$$

Taking the inner-product with $\mathcal{Z} \in \mathcal{H}$ in equation (5.12) and the consideration in equation (3.8), we reach that

$$0 = \Re < AU, \mathcal{Z} >= \mathcal{E}'(t) = -\widetilde{\kappa} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi.$$

$$\psi = 0,\ in\ \mathbb{R}.$$

Equation $(5.13)_2$ gives $\Phi(L) = 0$. We have to show that $\Phi = 0\ in\ (0, L)$.
Multiplying equation $(5.13)_1$ by $\overline{\Phi}$, integrating by parts, using the boundary conditions, we get

$$\int_0^L a|\Phi_x|^2 dx = 0.$$

And by Poincré's inequality , we get

$$\Phi = 0,\ in\ (0, L).$$

The proof is conclude. □

Lemma 5.3. *Let $\mathcal{A}$ be the operator defined by equation (3.5). Then we have*

- $i\mathbb{R} \subset \rho(\mathcal{A})\ if\ \eta \neq 0.$
- $i\mathbb{R}^* \subset \rho(\mathcal{A})\ if\ \eta = 0.$

Proof. We will prove that the operator $(i\varrho\mathcal{I} - \mathcal{A})$ is surjective, and this is divided in two cases.
Case 1: $\varrho \neq 0$ For this purpose , let $\mathcal{G} = (y_1, y_2)^T \in H$, we seek $\mathcal{Z} = (\Phi, \psi) \in \mathcal{D}(\mathcal{A})$, a solution of $(i\varrho\mathcal{I} - \mathcal{A})\mathcal{Z} = \mathcal{G}$, or equivalently we have to solve the system

$$\left\{ \begin{array}{l} i\varrho\Phi + i(a\Phi_x)_x = y_1, \\ i\varrho\psi + (\xi^2 + \eta)\psi - \mu(\xi)\Phi(L) = y_2. \end{array} \right. \tag{5.14}$$

From equation $(5.14)_2$, we get

$$\psi = \frac{\mu(\xi)\Phi(L)}{i\varrho + \xi^2 + \eta} + \frac{y_2}{i\varrho + \xi^2 + \eta}. \tag{5.15}$$

Set $\zeta = \widetilde{\kappa}\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{i\varrho + \xi^2 + \eta} d\xi$. Let $\mathcal{H}_1 = H^1_{a,L}(0, L)$ choose $\widetilde{\Phi} \in \mathcal{H}_1$. By multiplying equation $(5.14)_1$ by $\overline{\widetilde{\Phi}}$, we get

$$-\varrho\int_0^L \Phi\overline{\widetilde{\Phi}}dx + \int_0^L a\Phi_x\overline{\widetilde{\Phi}}dx + i\zeta\Phi(L)\overline{\widetilde{\Phi}(L)} = i\int_0^L y_1\overline{\widetilde{\Phi}}dx - i\overline{\widetilde{\Phi}(L)}\widetilde{\kappa}\int_{-\infty}^{+\infty} \frac{\mu(\xi)y_2}{i\varrho + \xi^2 + \eta}d\xi. \tag{5.16}$$

We can transform equation (5.16) as

$$\widetilde{\mathcal{B}}(\Phi, \widetilde{\Phi}) = \widetilde{\mathcal{B}}_1(\Phi, \widetilde{\Phi}) + \widetilde{\mathcal{B}}_2(\Phi, \widetilde{\Phi}) = \widetilde{\mathcal{L}}(\widetilde{\Phi}) \ \forall\widetilde{\Phi} \in \mathcal{H}_1, \tag{5.17}$$

where

$$\widetilde{\mathcal{L}}(\widetilde{\Phi}) = i\int_0^L y_1\overline{\widetilde{\Phi}}dx - i\overline{\widetilde{\Phi}(L)}\widetilde{\kappa}\int_{-\infty}^{+\infty} \frac{\mu(\xi)y_2}{i\varrho + \xi^2 + \eta}d\xi, \tag{5.18}$$

and

$$\begin{cases} \widetilde{\mathcal{B}}_1(\Phi, \widetilde{\Phi}) = & -\varrho\displaystyle\int_0^L \Phi\overline{\widetilde{\Phi}}dx, \\ \widetilde{\mathcal{B}}_2(\Phi, \widetilde{\Phi}) = & \displaystyle\int_0^L a\Phi_x\overline{\widetilde{\Phi}}dx + \zeta\Phi(L)\overline{\widetilde{\Phi}(L)}. \end{cases} \tag{5.19}$$

To achieve this proof we have to deal with two cases. The case when $\varrho < 0$ is a direct application of the Lax-Milgram lemme, the other case will be treated as follows.
Case: $\varrho > 0$ Let $\mathcal{H}_1'$ denote the dual space of $\mathcal{H}_1$, and define the following operators.

$$\widetilde{\mathcal{F}} : \mathcal{H}_1 \to \mathcal{H}_1', \tag{5.20}$$

$$\widetilde{\Phi} \to \widetilde{\mathcal{F}}(\widetilde{\Phi}). \tag{5.21}$$

$$\widetilde{\mathcal{F}}_i : \mathcal{H}_1 \to (\mathcal{H}_1)' \ i \in \{1, 2\}, \tag{5.22}$$

$$\widetilde{\Phi} \to \widetilde{\mathcal{F}}_i\widetilde{\Phi}. \tag{5.23}$$

Such that

$$\begin{cases} \widetilde{\mathcal{F}}(\widetilde{\Phi}) = \widetilde{\mathcal{B}}(\Phi, \widetilde{\Phi}) \ \forall\widetilde{\Phi} \in \mathcal{H}_1, \\ \widetilde{\mathcal{F}}_i(\widetilde{\Phi}) = \widetilde{\mathcal{B}}_i(\Phi, \widetilde{\Phi}) \ \forall\widetilde{\Phi} \in \mathcal{H}_1 \ \{1, 2\}. \end{cases} \tag{5.24}$$

From the definition of $\widetilde{\mathcal{B}}_2$ we can see that it is a sesquilinear continuous form and coercive on $\mathcal{H}_1 \times \mathcal{H}_1$. Then, from equation (5.19) and the Lax-Milgram lemma , the operator $\widetilde{\mathcal{B}}_2$ is an isomorphism.Furthermore $H^1_{a,L}(0, L)$ is compactly embedded in $L^2(0, L)$, which implies that $\widetilde{\mathcal{B}}_1$ is compact. Hence $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_1 + \widetilde{\mathcal{B}}_2$ is a Freedholm operator of index 0. Now, following the freedholm alternative, it remains to show that $\widetilde{\mathcal{F}}$ is injective to conclude that it is an isomorphism. Let $\widetilde{\Phi} \in \ker(\widetilde{\mathcal{F}})$ such that

$$\widetilde{\mathcal{F}}(\widetilde{\Phi}) = 0. \tag{5.25}$$

Taking $\widetilde{\Phi} = \Phi$ in equation (5.25) , we get:

$$\| \sqrt{a}\Phi_x\|^2_{L^2(0,L)} + i\zeta|\Phi(L)|^2 = \varrho\|\Phi\|^2_{L^2(0,L)}. \tag{5.26}$$

Then we have $\Phi(L) = 0$, and Lemma 5.2 will implies that $\ker(\widetilde{\mathcal{F}}) = 0$ .Furthermore, the Freedholm alternative ensures that $\widetilde{\mathcal{F}}$ is an isomorphism. The definition of $\widetilde{\mathcal{L}}$ implies that it is an anti-linear continuous form on $\mathcal{H}_1$. Consequently , equation (5.16) admits a unique solution on $\mathcal{H}_1$. Hence $i\varrho I - \mathcal{O}$ is surjective for all $\varrho \in \mathbb{R}^*$.

Case2: $\varrho = 0$ **and** $\eta \neq 0$ In yhe same way as before, let $\mathcal{G} = (y_1, y_2)^T \in H$, we seek $\mathcal{Z} = (\Phi, \psi) \in \mathcal{D}(\mathcal{A})$, a solution of $(i\varrho\mathcal{I} - \mathcal{A})\mathcal{Z} = \mathcal{G}$, or equivalently we have to solve the system

$$\begin{cases} i(a\Phi_x)_x = y_1, \\ (\xi^2 + \eta)\psi - \mu(\xi)\Phi(L) = y_2. \end{cases} \tag{5.27}$$

In this case, we can apply the Lax-Milgram lemma and conclude. Therefore, the proof is complete. □

Proof. Theorem 5.1
Following a general theorem due to Arent-Batty [8] and Lyubich-Vu [36] a C_0-semigroup $e^{\mathcal{A}t}$ is strongly stable if it does not have purely imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is at most countable, which are exactly the subject of Lemma 5.2 and Lemma 5.3, and we can conclude. □

5.2. **Polynomial Stability** $\eta \neq 0$.

Theorem 5.4. [12] *Let* $(\mathcal{P}(t))_{t>0}$ *a bounded* C_0*-semigroup on a Hilbert space* $\mathcal{Z}$*, with* $\mathcal{G}$ *as its generator. If*

$$i\mathbb{R} \in \rho(\mathcal{G}). \tag{5.28}$$

and $\exists l > 0$,

$$\overline{\lim_{|\omega|\longrightarrow\infty}} \|(i\omega\mathcal{I} - \mathcal{G})^{-1}\|_{\mathcal{L}(\mathcal{Z})} = O(|\omega|^l). \tag{5.29}$$

Then there is a $C > 0$, *such that :*

$$\|e^{\mathcal{G}t}U_0\|^2_{\mathcal{Z}} \leq \frac{C}{t^{\frac{2}{l}}}\|U_0\|^2_{\mathcal{D}(\mathcal{G})}. \tag{5.30}$$

We can now state the following result

Theorem 5.5. *The* C_0*-semigroup* $S_{\mathcal{A}}(t)$ *is polynomially stable and we have*

$$\|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{C}{t^{\frac{2}{3-2\alpha}}}\|U_0\|^2_{\mathcal{D}(\mathcal{A})}. \tag{5.31}$$

Proof. Without loss of generality that $|\varrho| > 1$. Taking Lemma 4.4 into account, the foregoing theorem is simplified to analyzing the resolvent equation. $(i\varrho\mathcal{I} - \mathcal{A})\mathcal{Z} = \mathcal{G}$, for $\varrho \in \mathbb{R}$, namely

$$\begin{cases} i\varrho\Phi + i(a\Phi_x)_x = y_1, \\ i\varrho\psi + (\xi^2 + \eta)\psi - \mu(\xi)\Phi(L) = y_2. \end{cases} \tag{5.32}$$

Step 1 Taking inner-product in $\mathcal{H}$ with $\mathcal{Z} = (\Phi, \psi)^T$ and making use of equation (3.8), we get

$$\left|\Re e < \mathcal{A}\mathcal{Z}, \mathcal{Z} >_{\mathcal{H}}\right| \leq \|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}}. \tag{5.33}$$

Equation (5.33) is equivalent to say

$$\widetilde{\kappa}\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi dx \leq \|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}}. \tag{5.34}$$

We have from the boundary conditions

$$\begin{aligned} |(a\Phi_x)(L)|^2 &\leq \widetilde{\kappa}^2\left(\int_{-\infty}^{+\infty} \mu(\xi)|\psi|d\xi\right)^2 \\ &\leq \widetilde{\kappa}^2\int_{-\infty}^{+\infty} \frac{\mu(\xi)^2}{(\xi^2 + \eta)}d\xi\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi \\ &\leq C\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}}. \end{aligned} \tag{5.35}$$

From equation $(5.32)_2$, we get

$$\Phi(L)\mu(\xi) = (i\varrho + \xi^2 + \eta)\psi - y_2, \qquad \forall\xi \in \mathbb{R}. \tag{5.36}$$

Multiplying equation (5.36) by $\dfrac{\mu(\xi)}{i\varrho + \xi^2 + \eta}$, we obtain

$$\frac{\Phi(L)\Phi^2(\xi)}{i\varrho + \xi^2 + \eta} = \mu(\xi)\psi - \frac{\mu(\xi)y_2}{i\varrho + \xi^2 + \eta}, \qquad \forall\xi \in \mathbb{R}. \tag{5.37}$$

Hence, taking absolute values of both sides of equation (5.37) and applying the triangle inequality, we obtain

$$\frac{|\Phi(L)|\Phi^2(\xi)}{|i\varrho + \xi^2 + \eta|} \leq \mu(\xi)|\psi| + \frac{\mu(\xi)|y_2(\xi)|}{|i\varrho + \xi^2 + \eta|}. \tag{5.38}$$

Integrating equation (5.38) over $\mathbb{R}$ with respect to the variable ξ, we get

$$|\Phi(L)|\int_{-\infty}^{+\infty} \frac{\Phi^2(\xi)}{|i\varrho + \xi^2 + \eta|}d\xi \leq \int_{-\infty}^{+\infty} \mu(\xi)|\psi|d\xi + \int_{-\infty}^{+\infty} \frac{\mu(\xi)|y_2(\xi)|}{|i\varrho + \xi^2 + \eta|}d\xi. \tag{5.39}$$

On the other hand, by applying the Cauchy-Schwartz inequality in (RHS) of equation (5.39), we deduce respectively

$$\int_{-\infty}^{+\infty} \mu(\xi)|\psi|d\xi \leq \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{(\xi^2 + \eta)}d\xi\right)^{\frac{1}{2}}, \tag{5.40}$$

and

$$\int_{-\infty}^{+\infty} \frac{\mu(\xi)|y_2(\xi)|}{|i\varrho + \xi^2 + \eta|}d\xi \leq \left(\int_{-\infty}^{+\infty} |y_2|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{|i\varrho + \xi^2 + \eta|^2}d\xi\right)^{\frac{1}{2}}. \tag{5.41}$$

Using equations (2.5), (2.6) and substituting equations (5.40) and (5.41) into equation (5.39), we get

$$c|\Phi(L)|(\varrho + \eta)^{\alpha - 1} \leq c\left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\psi|^2 d\xi\right)^{\frac{1}{2}} + c(\varrho + \eta)^{\frac{\alpha - 2}{2}}\left(\int_{-\infty}^{+\infty} |y_2|^2 d\xi\right)^{\frac{1}{2}}. \tag{5.42}$$

Taking the square of both sides and using the inequality $2AB \leq A^2 + B^2$, we obtain

$$(\varrho + \eta)^{2\alpha-2}|\Phi(L)|^2 \quad \leq 2c\left(\int_{-\infty}^{+\infty} (|\xi|^2 + \eta)|\psi|^2 d\xi\right) + 2c(\varrho + \eta)^{\alpha-2}\left(\int_{-\infty}^{+\infty} |y_2|^2 d\xi\right). \tag{5.43}$$

From equations (5.34) and (5.43), we can conclude that

$$|\Phi(L)|^2 \leq c|\varrho|^{2-2\alpha}\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}} + c\|\mathcal{G}\|^2_{\mathcal{H}}. \tag{5.44}$$

Step 2 In this step we have to estimate Φ, this will be done partially and we begin with the following lemma

Lemma 5.6. *Let ϕ be the function introduced in equation (5.5), encountering its properties given in equation (5.6). Then we have*

(1)

$$\chi = \varrho\int_0^L \phi_x|\Phi|^2 dx + \int_0^L a^2\left(\frac{\phi}{a}\right)_x |\Phi_x|^2 dx \leq c\left(|\varrho|^{3-2\alpha} + 1\right)\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}} + c|\varrho|\|\mathcal{G}\|^2_{\mathcal{H}}. \tag{5.45}$$

(2)

$$\|\Phi\|^2_{L^2(0,L)} \leq c\left(|\varrho|^{3-2\alpha} + 1\right)\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}} + c\left(|\varrho| + 1\right)\|\mathcal{G}\|^2_{\mathcal{H}}. \tag{5.46}$$

Proof. We are going to use some ideas from [46]. Multiplying equation $(5.32)_3$ by $\phi\overline{\Phi_x}$ and using equation $(5.32)_1$, we get

$$\varrho\int_0^L \phi\overline{\Phi_x}\Phi dx + \int_0^L \phi\overline{\Phi_x}(a\Phi_x)_x dx = -i\int_0^L y_1\phi\overline{\Phi_x}dx. \tag{5.47}$$

We deduce from equation (5.47) that

$$\varrho\int_0^L \phi\frac{d}{dx}|\Phi|^2 dx + \int_0^L \left(\frac{\phi}{a}\right)\frac{d}{dx}|a\Phi_x|^2 dx = 2i\int_0^L y_1\phi\overline{\Phi_x}dx. \tag{5.48}$$

Taking the real part of equation (5.48), and performing an integration by parts in the right-hand side (RHS) to obtain

$$\chi = \Re e\left(2i\int_0^L y_1\phi\overline{\Phi_x}dx + \varrho\phi(L)|\Phi(L)|^2 + \left(\frac{\phi}{a}\right)(L)|(a\Phi_x)(L)|^2\right). \tag{5.49}$$

Now using Cauchy-Schwartz inequality in the firs term of (RHS) of equation (5.49), the estimate in equations (5.35) and (5.44), we get

$$\chi \leq c|\varrho|^{3-2\alpha}\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}} + c|\varrho|\|\mathcal{G}\|^2_{\mathcal{H}} + \|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}}, \tag{5.50}$$

and this proves one. For the second estimation, we are going to use Poincaré's inequality, the assumption that $|\varrho| > 1$ and the properties of ϕ listed in equation (5.6)

$$\begin{aligned}\varrho\int_0^L \phi_x|\Phi|^2 dx + \int_0^L a^2\left(\frac{\phi}{a}\right)_x |\Phi_x|^2 dx \quad &\geq \int_0^L \phi_x|\Phi|^2 dx + \int_0^L a|\Phi_x|^2 dx \\ &\geq C\|\Phi\|^2_{L^2(0,L)}.\end{aligned} \tag{5.51}$$

Now we can combine equations (5.50) and (5.51) in one equation and get

$$\|\Phi\|^2_{L^2(0,L)} \leq c\left(|\varrho|^{3-2\alpha} + 1\right)\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}} + c|\varrho|\|\mathcal{G}\|^2_{\mathcal{H}},$$

this concludes the proof. □

Step 3 From equation (5.34), we deduce

$$\|\psi\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} |\psi|^2 d\xi \le C\int_{-\infty}^{+\infty} (|\xi|^2+\eta)|\psi|^2 \le C\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}}. \tag{5.52}$$

The combination of equation (5.46) and equation (5.52), gives us

$$\|\mathcal{Z}\|^2_{\mathcal{H}} \le c\left(|\varrho|^{3-2\alpha}+1\right)\|\mathcal{Z}\|_{\mathcal{H}}\|\mathcal{G}\|_{\mathcal{H}} + c\left(|\varrho|+1\right)\|\mathcal{G}\|^2_{\mathcal{H}}. \tag{5.53}$$

This implies that

$$\|\mathcal{Z}\|_{\mathcal{H}} \le C|\varrho|^{3-2\alpha}\|\mathcal{G}\|_{\mathcal{H}}. \tag{5.54}$$

From the above inequality we can directly conclude. □

6. Conclusion and Outlook

In our investigation, we studied the boundary stabilization of the non-degenerate Schrödinger equation using a dissipation law of fractional integral type. Through spectral analysis, we showed that the stability is not uniform. To establish strong asymptotic stability, we utilized the Arendt-Batty Theorem. Additionally, in the presence of $\eta > 0$, we derived polynomial energy decay rates that depend on α. Three factors make our work significant: first, and most crucially, the nature of the equation, which hasn't received much attention in the literature. Second is the boundary stabilization, which is new and was introduced only recently in [38], and lastly is the use of the theorem 3.2, which is not common in most of the papers. In the future, we aim to explore similar systems, focusing on the following areas as well:

$$\left\{\begin{array}{ll} y_t(x,t) - i\Delta y(x,t) = 0, & x\in\Omega, t>0,\\ y(x,0) = y_0(x,t), & x\in\Omega, t>0,\\ y(x,t) = 0, & x\in\Gamma_0, t>0,\\ \frac{\partial y}{\partial \nu} = i\mu_1\mathcal{I}^{1-\alpha,\eta}y(x,t) + i\mu_2\mathcal{I}^{1-\alpha,\eta}y(x,t-\tau), & \in\Gamma_1, t>0,\\ y(x,t-\tau) = f_0(x,t-\tau), & x\in\Omega, 0<t<\tau, \end{array}\right.$$

and

$$\left\{\begin{array}{ll} y_t(x,t) = i\Delta y(x,t) + \{i\mu_1 y(x,t) + i\mu_2\mathcal{I}^{1-\alpha,\eta}y(x,t-\tau)\}, & x\in\Omega, t>0\\ y(x,0) = y_0(x,t), & x\in\Omega, t>0,\\ y(x,t) = 0, & x\in\Gamma, t>0,\\ y(x,t-\tau) = g_0(x,t-\tau), & x\in\Omega, 0<t<\tau, \end{array}\right.$$

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