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SOME RESULTS ON THE DUAL SMALE'S CONJECTURE

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Abstract. Let p be a nonlinear complex polynomial in one variable. This paper explores a property of the extremal polynomials of the dual Smale conjecture and demonstrates the conditions under which the dual Smale conjecture holds in a special case.

1. Introduction and Main Results

If $p(z)$ is a polynomial of degree $n \geq 2$, then $p(z)$ has $n-1$ critical points counting multiplicities. A complex number $\zeta \in \mathbb{C}$ is called a critical point of $p(z)$ if $p'(\zeta) = 0$. Denote by $z_1, z_2, \ldots, z_{n-1}$ the critical points of $p(z)$. For any non-critical point $z \in \mathbb{C}$, in [10], Smale proved that

$$\min_j \left| \frac{p(z) - p(z_j)}{(z - z_j)p'(z)} \right| \leq 4 \tag{1.1}$$

It is conjectured that the constant 4 in Equation (1.1) can be replaced by $\frac{n-1}{n}$. This conjecture is known as Smale's Mean Value Conjecture.

If we denote

$$S(p) = \min_j \left| \frac{p(z) - p(z_j)}{(z - z_j)p'(z)} \right|$$

then the Smale's Mean Value Conjecture can be stated as

$$\sup S(p) = \frac{n-1}{n},$$

when the supremum is chosen for any polynomial $p(z)$ with degree n.

Since $S(p)$ is invariant under pre- and post- compositions of affine maps,we can normalize $p(z)$ with $p(0) = 0$ and $p'(0) = 1$, then

$$S(p) = \min_\zeta \left\{ \left| \frac{p(\zeta)}{\zeta} \right| : p'(\zeta) = 0,\ p(0) = 0,\ p'(0) = 1 \right\}.$$

Let

$$K_n = \sup\{S(p) : \deg p = n,\ p(0) = 0,\ p'(0) = 1\}.$$

Then, Smale's Mean Value Conjecture is simplified to $K_n = 1 - \frac{1}{n}$.

Significant progress has been made in relation to Smale's Mean Value Conjecture. As shown in [11], the case for $n = 4$ can be proven based on Tischer's research results.

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When p has only real critical points, Sheil-Small [9] proved that $S(p) < e-2$. Later, Rahman and Schmeisser [8] improved this result by showing that for $n \geq 3$, thereby lowering the upper bound of $S(p)$.

In 2006, Crane [1] introduced $K(n)$ as the supremum of $S(p)$ across all non-linear polynomials p of degree n with $p'(0) \neq 0$. A polynomial that achieves this supremum, where $S(p) = K(n)$, is called an extremal polynomial. The sequence $\{K(n)\}$ is monotonically increasing, with $K(n+1) \geq K(n)$ for all $n \geq 2$. In 2019, the authors [6] verified the conjecture for $n = 6$. Recently, Hinkkanen et al. [7] proved the conjecture for $n = 7$.

The Smale's mean value conjecture has been proven in certain special cases; for example, when all critical points of p lie on a line through the origin, a sharper uniform bound $S(p) \leq e - 2$ holds; see [8], Theorem 7.2.5

In addition, there are similar mean value and dual conjectures for finite Blaschke products. If we let M_n be the counterpart of $K(n)$ for finite Blaschke products, Ng and Zhang [7] showed that

$$M_n \leq 2 \cdot \frac{2n - 1 + (2n-3)4^{\frac{1}{n-1}}}{2n-1}.$$

And Dubinin [4] pointed out that there exists a critical point ζ satisfying the smale mean value conjecture.

Parallel to Smale's Mean Value Conjecture, Dubinin and Sugawa [2] proposed the Dual Smale's Conjecture as follows:

Let

$$T(p) = \max_{\zeta} \left\{ \left| \frac{p(\zeta)}{\zeta} \right| : p'(\zeta) = 0,\ p(0) = 0,\ p'(0) = 1 \right\},$$

$$L_n = \inf\{T(p) : \deg p = n,\ p(0) = 0,\ p'(0) = 1\}.$$

The Dual Smale Conjecture states that $L_n = \frac{1}{n}$.

Research on the Dual Smale Conjecture has also achieved some results. Dubinin and Sugawa [3] pointed out that there exists a critical point ζ that satisfies the relevant conditions. In 2016, Ng and Zhang [7] showed that $L_n \geq \frac{1}{4^n}$ for finite Blaschke products. In 2019, Dubinin [2] derives a conclusion regarding the Dual Smale Conjecture.The author mentioned that when n is sufficiently large, a certain term in this condition can be replaced by $\frac{1}{n}$, thereby obtaining a larger lower bound.

In this paper, we investigate a property of extremal polynomials related to the Dual Smale Conjecture, and we also examine a special case in which the conjecture holds. Furthermore, we prove that for $n \geq 2$, $p(0) = 0$, and $p'(0) \neq 0$, there exists a solution to the expression $\max\left\{ \left| \frac{p(\zeta)}{\zeta p'(0)} \right| : p'(\zeta) = 0 \right\}$.

1.1. **Main Results.**

Theorem 1.1. *For each $n \geq 2$, we can observe that*

$$L(n+1) \leq L(n).$$

Theorem 1.2. *Let* p *be a polynomial of degree* $n \geq 2$ *with* $p(0) = 0$ *and* $p'(0) \neq 0$. *Suppose that the critical points of* p *are all real. Then*

$$\max\left\{\left|\frac{p(\zeta)}{\zeta p'(0)}\right| : p'(\zeta) = 0\right\} \geq \frac{1}{n},$$

the first bound being $1/2$ *when* $n = 2$. *Equality is attained throughout if* $n = 2$, *or if* $n = 3$ *and* $p''(0) = 0$. *For* $n > 3$, *the last inequality is strict.*

2. Proof

2.1. Proof of Theorem 1.

Proof of Theorem 1. Let $L(n)$ denote the greatest lower bound of $T(p)$. For any given $\epsilon > 0$, consider a polynomial $p \in \mathcal{G}_N$ such that $T(p) < L(n) + \epsilon$. We construct a polynomial $r(z)$ of degree $n + 1$ parameterized by a nonzero constant ζ via the integral

$$r(z) = \int_0^z \left(1 - \frac{w}{\zeta}\right) p'(w)\, dw.$$

Note that $r(0) = 0$, $r'(0) = p'(0) = 1$, and the critical points of r are those of p together with ζ. If we denote the objective value at a critical point ζ_i of p as λ_i, then its objective value for r is

$$\frac{r(\zeta_i)}{\zeta_i} = \frac{1}{\zeta_i}\int_0^{\zeta_i} \left(1 - \frac{w}{\zeta}\right) p'(w)\, dw = \lambda_i - \frac{1}{\zeta}\left(\frac{1}{\zeta_i}\int_0^{\zeta_i} wp'(w)\, dw\right).$$

λ_i is the objective value of p. As $|\zeta| \to \infty$, we have

$$\frac{1}{\zeta}\left(\frac{1}{\zeta_i}\int_0^{\zeta_i} wp'(w)\, dw\right) \to 0.$$

From the previous steps, we obtain the inequality for $T(r)$:

$$T(r) < T(p) + \epsilon.$$

Since we know $T(p) < L(n) + \varepsilon$ (from the given conditions in our context), we can substitute this in:

$$T(r) < T(p) + \varepsilon < L(n) + \varepsilon + \varepsilon = L(n) + 2\varepsilon.$$

Now, recall that $L(n+1)$ is defined as the greatest lower bound of S for polynomials of degree $n + 1$. Since r is a polynomial of degree $n + 1$, the greatest lower bound $L(n + 1)$ must satisfy:

$$L(n+1) \leq T(r).$$

Combining this with the inequality we just derived for $T(r)$:

$$L(n+1) < L(n) + 2\varepsilon.$$

But this holds for any $\varepsilon > 0$. If we take the limit as $\varepsilon \to 0^+$, we can conclude that:

$$L(n+1) \leq L(n).$$

□

2.2. **Proof of Theorem 2.**

Lemma 2.1. *Let g be a polynomial of positive degree at most n and with only real zeros. Suppose that, on an interval $[a,b]$, it increases monotonically from 0 to $g(b)>0$. Then*

$$\int_a^b g(x)\,dx \geq (b-a)g(b)\cdot\frac{1}{n+1}, \tag{*}$$

where the right-hand side is $(b-a)g(b)/2$ when $n=1$.

Proof. Omitting a trivial case, we may suppose that the degree of g is at least two. Moreover, by an affine transformation, we may restrict ourselves to the situation where $a=0$, $b=1$, and $g(b)=1$. Then the polynomials under consideration may be written as

$$g(x)=x\prod_{\kappa=1}^{k}\frac{x-\alpha_\kappa}{1-\alpha_\kappa}\prod_{\lambda=1}^{\ell}\frac{\beta_\lambda-x}{\beta_\lambda}, \tag{2.1}$$

where

$$k+\ell\leq n-1,\ \alpha_\kappa<0\,(\kappa=1,\dots,k),\ \beta_\lambda>1\,(\lambda=1,\dots,\ell). \tag{2.2}$$

Such a polynomial is monotonic on $[0,1]$ if and only if $g'(1)\geq 0$, or equivalently,

$$\sum_{\lambda=1}^{\ell}\frac{1}{\beta_\lambda-1}\leq 1+\sum_{\kappa=1}^{k}\frac{1}{1-\alpha_\kappa}. \tag{2.3}$$

We now consider the value of g at $x\in(0,1)$ as a function of $\alpha_1,\dots,\alpha_k$ and $\beta_1,\dots,\beta_\ell$, and maximize it under the side conditions (2.2) and (2.3). The following observations can be verified by elementary calculus.

Step 1. For $x\in(0,1)$, the value $g(x)$ decreases when the numbers α_κ and β_λ increase. However, they can increase only until equality occurs in (2.3).

Step 2. Suppose that equality holds in (2.3). Keep all α_κ and β_λ fixed, except for a pair of distinct numbers α_{κ_1} and α_{κ_2} if there is any. Vary α_{κ_1} and α_{κ_2} such that equality in (2.3) is preserved. Then $g(x)$ decreases if the distance between α_{κ_1} and α_{κ_2} increases. An analogous observation holds for a pair of distinct numbers β_{λ_1} and β_{λ_2}. Using this, we conclude that the polynomial which is largest on $(0,1)$ is amongst those of the form

$$g(x):=x\left(\frac{x-\alpha}{1-\alpha}\right)^k\left(\frac{\beta-x}{\beta}\right)^\ell, \tag{2.4}$$

where $k+\ell\leq n-1$, $\alpha<0$, $\beta>1$, and

$$\frac{\ell}{\beta-1}=1+\frac{k}{1-\alpha}.$$

Step 3. If we vary α and β such that (2.4) is preserved, then $g(x)$ decreases as $\alpha\to 0$ and $\beta\to\infty$. In particular, $g(x)$ converges to

$$g_e(x):=x^{k+1}. \tag{2.5}$$

Step 4. For the polynomials (2.5), we verify that

$$g_e(x)>g_m(x)\quad(x\in(0,1),\ \ell<m).$$

This shows that, for any polynomial (2.1), subject to (2.2) and (2.3), we have

$$\int_0^1 g(x)\,dx \geq \int_0^1 g_{n-1}(x)\,dx.$$

Calculating the integral on the right-hand side, we find that inequality (*) holds with $a = 0$, $b = 1$, and $g(b) = 1$. □

Theorem 2.2. *Let p be a polynomial of degree $n \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$. Suppose that the critical points of p are all real. Then*

$$\max\left\{\left|\frac{p(\zeta)}{\zeta p'(0)}\right| : p'(\zeta) = 0\right\} \geq \frac{1}{n},$$

the first bound being $1/2$ when $n = 2$. Equality is attained throughout if $n = 2$, or if $n = 3$ and $p''(0) = 0$. For $n > 3$, the last inequality is strict.

Proof. As an admissible normalization, we may suppose that $p'(0) = 1$. Then p' is real-valued on the real line. It is clear, by considering the graph of p, that there exists a critical point ζ, neighboring the origin, so that p' is monotonic on the interval with endpoints 0 and ζ. Replacing x by $-x$ when $\zeta < 0$, we arrive at a situation which allows us to apply Lemma 2.1 with n replaced by $n - 1$. We conclude that

$$\left|\frac{p(\zeta)}{\zeta p'(0)}\right| = \left|\frac{1}{\zeta}\int_0^\zeta p'(x)\,dx\right| = \frac{1}{\zeta}\int_0^\zeta p'(x)\,dx \geq \frac{1}{n}.$$

□

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