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THE QUENCHING FOR A NONLINEAR DIFFUSION EQUATION WITH NONLINEAR SINGULAR BOUNDARY CONDITION: NUMERICAL APPROXIMATION

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Abstract. This paper is concerned with the study of the numerical approximation for the following nonlinear diffusion equation $\dfrac{\partial \Psi(u)}{\partial t} = u_{xx} + f(x)(1-u)^{-p}$, $0 < x < 1$, $t > 0$, with a nonlinear singular boundary flux $u_x(0,t) = 0$, $u_x(1,t) = -g(u(1,t))$, $t > 0$ and an initial solution $u(x,0) = u_0(x)$, $0 \le x \le 1$. We use the finite differences method to obtain the semidiscrete scheme. We find some conditions under which the solution of the semidiscrete form obtained quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

1. Introduction

We consider the nonlinear diffusion equation with nonlinear singular boundary flux

$$\frac{\partial \Psi(u)}{\partial t} = u_{xx} + f(x)(1-u)^{-p}, \quad 0 < x < 1, \ t > 0, \tag{1.1}$$

$$u_x(0,t) = 0, u_x(1,t) = -g(u(1,t)), \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{1.3}$$

where Ψ is an appropriately smooth, concave and strictly monotonically increasing function which satisfies $\Psi(0) = 0$, $\Psi(1) = 1$, g satisfies $g(s) > 0$, $g'(s) < 0$, $g''(s) \ge 0$, for $s > 0$, $\lim_{s \to 0^+} g(s) = +\infty$, f is a continuous function such that $f(x) > 0$, $f'(x) < 0$, $u_0 : [0,1] \longrightarrow (0,1)$ is smooth enough and nonincreasing and p is a positive constant.

When $\Psi(u) = u^m$, the problem (1.1)–(1.3) is known as the classical porous medium equation which shows a number of physical phenomenon in the nature such as the flow of an isentropic gas through a porous medium ([7], [10]) and heat transfer or diffusion [13].

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Definition 1.1. We say that the solution u of the problem (1.1)–(1.3) quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_q)$ but $\lim\limits_{t\to T_q} \|u(.,t)\|_\infty = 1$, where $\|u(.,t)\|_\infty = \max\limits_{0\le x\le 1} |u(x,t)|$. The time T_q is called quenching time of the solution u.

The problem (1.1)–(1.3) has been a subject of investigation by the authors of [6]. They show that the solution u of this problem quenches in a finite time and the time derivative blows up at a quenching point. They finish their study by estimating the lower bound of the quenching time.

Before them, Yang [14] studies the quenching phenomenon of the following nonlinear diffusion equation with nonlinear source and singular boundary flux

$$\frac{\partial A(u)}{\partial t} = u_{xx} + (1-u)^{-\alpha}, \quad 0 < x < 1, \ t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, u_x(1,t) = -B(u(1,t)), \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{1.6}$$

with A and B having respectively the same properties as the functions Ψ and g of (1.1)–(1.3) and $\alpha > 0$. He shows that the solution u of (1.4)–(1.6) quenches in finite time T_q and $x = 0$ is the unique quenching point. He also shows that the time derivative u_t blows up at the quenching point and he gives a lower bound of the quenching time.

Other particular cases of problem (1.1)–(1.3) were also the subject of investigation (see [4], [12] and references therein). However the numerical study remains unexploited. We will therefore investigate the case of a semidiscrete approximation. For this, we will refer to the following works ([2], [3], [5], [8], [11],...).

The problem (1.1)–(1.3) may be rewritten in the following model

$$u_t = \alpha(u)u_{xx} + \alpha(u)f(x)(1-u)^{-p}, \quad 0 < x < 1, \ t > 0, \tag{1.7}$$

$$u_x(0,t) = 0, u_x(1,t) = -g(u(1,t)), \quad t > 0, \tag{1.8}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{1.9}$$

where $\alpha(u) = \dfrac{1}{\Psi'(u)}$.

Let $I \ge 2$ be an integer and let $k = \dfrac{1}{I}$. Define the grid $x_i = ik$, $0 \le i \le I$ and approximate the solution u of problem (1.7)–(1.9) by the solution $U_k(t) = (U_0(t), U_1(t), ..., U_I(t))^T$ and the initial condition u_0 in (1.9) by the initial condition $\varphi_k = (\varphi_0, \varphi_1, ..., \varphi_I)^T$ of the following semidiscrete equations

$$\frac{dU_0(t)}{dt} = \alpha(U_0(t))\delta^2 U_0(t) + f_0\alpha(U_0(t))(1-U_0(t))^{-p}, \tag{1.10}$$

$$\frac{dU_i(t)}{dt} = \alpha(U_i(t))\delta^2 U_i(t) + f_i\alpha(U_i(t))(1-U_i(t))^{-p}, \quad 1 \le i \le I-1, \tag{1.11}$$

$$\frac{dU_I(t)}{dt} = \alpha(U_I(t))\delta^2 U_I(t) - \frac{2\alpha(U_I(t))}{k} g(U_I(t)) + f_I\alpha(U_I(t))(1-U_I(t))^{-p}, \tag{1.12}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \tag{1.13}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{k^2}, \quad 1 \le i \le I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{k^2}, \qquad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{k^2},$$

$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{k}, \qquad \delta^+ \varphi_i \le 0, \quad 0 \le i \le I-1,$$

$\alpha(U_i(t))$ is an approximation of $\alpha(u(x_i,t))$ and f_i is an approximation of $f(x_i)$, $0 \le i \le I$.

In the next section, we present some lemmas which will be used throughout the paper. In section 3, we study the convergence of the semidiscrete solution. In section 4, under some conditions, we prove that the solution of the semidiscrete form of (1.7)–(1.9) quenches in a finite time and we study the convergence of the numerical quenching time. Finally, in last section, we give some numerical experiments.

2. Properties of the semidiscrete solution

In this section, we give some lemmas which will be used later.

Lemma 2.1. *Let* $\chi_k(t) \in C^0([0,T), \mathbb{R}^{I+1})$, $\varrho_k(t) \in C^0([0,T), \mathbb{R}_+^{I+1})$ *and* $V_k(t) \in C^1([0,T), \mathbb{R}^{I+1})$ *such that*

$$\frac{dV_i(t)}{dt} - \varrho_i(t)\delta^2 V_i(t) + \chi_i(t) V_i(t) \ge 0, \quad 0 \le i \le I, \ t \in [0,T),$$

$$V_i(0) \ge 0, \quad 0 \le i \le I.$$

Then $V_i(t) \ge 0$, $0 \le i \le I$, $t \in [0,T)$.

Proof. Let $T_0 < T$. Define the vector $\Phi_k(t) = e^{\lambda t} V_k(t)$ where λ is such that $\chi_i(t) - \lambda > 0$ for $t \in [0,T_0]$, $0 \le i \le I$.

Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} \Phi_i(t)$. $\forall i \in \{0,...,I\}$, $\Phi_i(t)$ is continous on the compact $[0,T_0]$; there exists $i_0 \in \{0,...,I\}$ and $t_0 \in [0,T_0]$ such that $m = \Phi_{i_0}(t_0)$.

We observe that:

$$\frac{d\Phi_{i_0}(t_0)}{dt} = \lim_{h \to 0} \frac{\Phi_{i_0}(t_0) - \Phi_{i_0}(t_0 - h)}{h} \le 0, \quad 0 \le i_0 \le I, \tag{2.1}$$

$$\delta^2 \Phi_{i_0}(t_0) = \frac{\Phi_{i_0+1}(t_0) - 2\Phi_{i_0}(t_0) + \Phi_{i_0-1}(t_0)}{k^2} \ge 0, \quad 1 \le i_0 \le I-1, \tag{2.2}$$

$$\delta^2 \Phi_{i_0}(t_0) = \frac{2\Phi_1(t_0) - 2\Phi_0(t_0)}{k^2} \ge 0, \quad i_0 = 0, \tag{2.3}$$

$$\delta^2 \Phi_{i_0}(t_0) = \frac{2\Phi_{I-1}(t_0) - 2\Phi_I(t_0)}{k^2} \ge 0, \quad i_0 = I. \tag{2.4}$$

Moreover, by a straightforward computation, we get

$$\frac{d\Phi_{i_0}(t_0)}{dt} - \varrho_{i_0}(t_0)\delta^2 \Phi_{i_0}(t_0) + (\chi_{i_0}(t_0) - \lambda)\Phi_{i_0}(t_0) \ge 0. \tag{2.5}$$

Using (2.1)–(2.4), we deduce from (2.5) that $(\chi_{i_0}(t_0)-\lambda)\Phi_{i_0}(t_0)\geq 0$, which implies that $\Phi_{i_0}(t_0)\geq 0$. We deduce that $V_k(t)\geq 0$, $\forall t\in[0,T_0]$ and the proof is complete. □

Lemma 2.2. *Let $V_k(t)$, $W_k(t)\in C^1([0,T),\mathbb{R}^{I+1})$ and $j\in C^0(\mathbb{R},\mathbb{R})$ such that $\forall t\in[0,T)$ and $0\leq i\leq I$,*

$$\frac{dV_i(t)}{dt}-\alpha(V_i(t))\delta^2V_i(t)+j(V_i(t))<\frac{dW_i(t)}{dt}-\alpha(W_i(t))\delta^2W_i(t)+j(W_i(t)),\quad(2.6)$$

$$V_i(0)<W_i(0).\quad(2.7)$$

Then $V_i(t)<W_i(t)$, $0\leq i\leq I$, $t\in[0,T]$.

Proof. Let $X_k(t)$ a vector such that $X_i(t)=W_i(t)-V_i(t)$ and let t_0, be the first $t>0$ such that $X_{i_0}(t)>0$, $\forall t\in[0,t_0)$ but $X_{i_0}(t_0)=0$ for a certain $i_0\in\{0,...,I\}$.

We observe that:

$$\frac{dX_{i_0}(t_0)}{dt}=\lim_{h\to 0}\frac{X_{i_0}(t_0)-X_{i_0}(t_0-h)}{h}\leq 0,\quad 0\leq i_0\leq I,$$

$$\delta^2X_{i_0}(t_0)\geq 0,\quad 0\leq i_0\leq I.$$

Which implies that

$$\frac{dX_{i_0}(t_0)}{dt}-\alpha(V_{i_0}(t_0))\delta^2X_{i_0}(t_0)-\alpha'(\theta_{i_0}(t_0))X_{i_0}(t_0)\delta^2W_{i_0}(t_0)+j(W_{i_0}(t_0))-j(V_{i_0}(t_0))\leq 0$$

and

$$\frac{dW_i(t)}{dt}-\alpha(W_i(t))\delta^2W_i(t)+j(W_i(t))\leq\frac{dV_i(t)}{dt}-\alpha(V_i(t))\delta^2V_i(t)+j(V_i(t)),$$

where θ_{i_0} is an intermediate value between V_{i_0} and W_{i_0}. This inequality contradicts (2.6) which ends the proof. □

Lemma 2.3. *Let U_k be the solution of (1.10)–(1.13). Then we have for $t\in[0,T_q^k)$ and $0\leq i\leq I-1$,*

$$U_i(t)>U_{i+1}(t).$$

Proof. Introduce the vector $Y_k(t)$ such that $Y_i(t)=U_i(t)-U_{i+1}(t)$ for $t\in(0,T_q^k)$, $i\in\{0,...,I-1\}$. Let t_0, be the first $t>0$ such that $Y_{i_0}(t)>0$, $\forall t\in[0,t_0)$ but $Y_{i_0}(t_0)=0$ for a certain $i_0\in\{0,...,I-1\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that

$$\frac{dY_{i_0}(t_0)}{dt}=\lim_{h\to 0}\frac{Y_{i_0}(t_0)-Y_{i_0}(t_0-h)}{h}\leq 0,\quad 0\leq i_0\leq I-1,$$

$$\delta^2Y_{i_0}(t_0)\geq 0,\quad 0\leq i_0\leq I-1.$$

Moreover, by a straightforward computation, we get

$$\frac{dY_{i_0}(t_0)}{dt}-\alpha(U_{i_0}(t_0))[\delta^2Y_{i_0}(t_0)+(f_{i_0}-f_{i_0+1})(1-U_{i_0}(t_0))^{-p}+pf_{i_0+1}(1-U_{i_0}(t_0))^{-p-1}Y_{i_0}(t_0)]$$
$$+\alpha'(\varpi_{i_0}(t_0))\left[\delta^2U_{i_0+1}(t_0)-f_{i_0+1}(1-U_{i_0+1}(t_0))^{-p}\right]Y_{i_0}(t_0)<0,0\leq i_0\leq I-2,$$
$$\frac{dY_{I-1}(t_0)}{dt}-\alpha(U_{I-1}(t_0))\left[\delta^2Y_{I-1}(t_0)+\frac{2}{k}g(U_I)+(f_{I-1}-f_I)(1-U_I(t_0))^{-p}\right]$$

$$-p\alpha(U_{I-1}(t_0))f_{I-1}(1-U_I(t_0))^{-p-1}Y_{I-1}(t_0)+\alpha'(\varpi_I(t_0))\delta^2U_{I-1}(t_0)Y_{I-1}(t_0)$$
$$+\alpha'(\varpi_I(t_0))f_{I-1}(1-U_{I-1}(t_0))^{-p}Y_{I-1}(t_0)<0,$$

where ϖ_{i_0} is an intermediate value between U_{i_0} and U_{i_0+1}. But these inequalities contradict (1.10)–(1.12) and this proof is complete. □

Lemma 2.4. *Let U_k be the solution of (1.10)–(1.13). Then we have*

$$\frac{dU_i(t)}{dt}\geq 0,\quad 0\leq i\leq I,\ t\in(0,T_q^k).$$

Proof. Consider the vector $Z_k(t)$ such that $Z_i(t)=\frac{dU_i(t)}{dt}$, $t\in(0,T_q^k)$, $i\in\{0,...,I\}$. Let t_0, be the first $t\in(0,T_q^k)$ such that $Z_{i_0}(t)\geq 0$, $\forall t\in[0,t_0)$ but $Z_{i_0}(t_0)<0$ for a certain $i_0\in\{0,...,I\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the inequality above. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{h\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-h)}{h}<0,\quad 0\leq i_0\leq I,$$
$$\delta^2Z_{i_0}(t_0)\geq 0,\quad 0\leq i_0\leq I.$$

Moreover, by a straightforward computation, we get

$$\frac{dZ_{i_0}(t_0)}{dt}-\left(\delta^2U_{i_0}(t_0)+f_{i_0}(1-U_{i_0}(t_0))^{-p}\right)\alpha'(U_{i_0}(t_0))Z_{i_0}(t_0)-$$
$$(\delta^2Z_{i_0}(t_0)+pf_{i_0}(1-U_{i_0}(t_0))^{-p-1}Z_{i_0}(t_0))\alpha(U_{i_0}(t_0))<0,0\leq i_0\leq I-1,$$
$$\frac{dZ_I(t_0)}{dt}-\left(\delta^2U_I(t_0)+f_I(1-U_I(t_0))^{-p}-\frac{2}{k}g(U_I(t_0))\right)\alpha'(U_I(t_0))Z_I(t_0)$$
$$-\left(\delta^2Z_I(t_0)+pf_I(1-U_I(t_0))^{-p-1}Z_I(t_0)-\frac{2}{k}g'(U_I(t_0))Z_I(t_0)\right)\alpha(U_I(t_0))<0,$$

but these inequalities contradict (1.10)–(1.12) and this proof is complete. □

Lemma 2.5. *Let $U_k\in\mathbb{R}^{I+1}$ such that $\|U_k\|_\infty<1$ and let p be a positive constant. Then, we have*

$$\delta^2(1-U_i)^{-p}\geq p(1-U_i)^{-p-1}\delta^2U_i,\quad 0\leq i\leq I.$$

Proof. Let us introduce $f(s)=(1-s)^{-p}$. We observe that f is a convex function for nonnegative values of s. Apply Taylor's expansion to obtain

$$\delta^2f(U_0)=f'(U_0)\delta^2U_0+\frac{(U_1-U_0)^2}{h^2}f''(\theta_0).$$
$$\delta^2f(U_i)=f'(U_i)\delta^2U_i+\frac{(U_{i+1}-U_i)^2}{2h^2}f''(\theta_i)+\frac{(U_{i-1}-U_i)^2}{2h^2}f''(\eta_i),\quad 1\leq i\leq I-1.$$
$$\delta^2f(U_I)=f'(U_I)\delta^2U_I+\frac{(U_{I-1}-U_I)^2}{h^2}f''(\eta_I).$$

where θ_i is an intermediate between U_i and U_{i+1} and η_i the one between U_{i-1} and U_i. We use the fact that $\|U_k\|_\infty<1$ to complete the proof. □

3. Convergence of Semidiscrete Solution

Theorem 3.1. *Assume that the problem (1.7)–(1.9) has a solution* $u \in C^{4,1}([0,1]\times[0,T])$ *such that* $\sup_{t\in[0,T]} \|u(.,t)\|_\infty = \zeta < 1$. *Suppose that the initial data at (1.13) verifies*

$$\|\varphi_k - u_k(0)\|_\infty = o(1) \quad as \quad k \longrightarrow 0. \tag{3.1}$$

Then, for k small enough, the semidiscrete problem (1.10)–(1.13) has a unique solution $U_k \in C^1([0,T],\mathbb{R}^{I+1})$ *such that*

$$\max_{t\in[0,T]} \|U_k(t) - u_k(t)\|_\infty = O(\|\varphi_k - u_k(0)\|_\infty + k) \quad as\ k \longrightarrow 0,$$

where $T < \min\{T_q; T_q^k\}$.

Proof. Since $u \in C^{4,1}([0,1]\times[0,T])$, there exists a positive constant ξ such that

$$\frac{\|u_{xxx}\|_\infty}{3} \le \xi \quad \text{and} \quad \frac{\|u_{xxxx}\|_\infty}{12} \le \xi. \tag{3.2}$$

The problem (1.10)–(1.13) has for each k, a unique solution $U_k \in C^1([0,T],\mathbb{R}^{I+1})$. Let $t(k) \le T$ the greatest value of $t > 0$ such that there exists a positive constant β (with $\zeta < \beta < 1$) such that

$$\|U_k(t) - u_k(t)\|_\infty < \frac{\beta-\zeta}{2} \quad \text{for} \quad t \in (0, t(k)). \tag{3.3}$$

The relation (3.1) implies that $t(k) > 0$ for k small enough. By the triangular inequality, we obtain

$$\|U_k(t)\|_\infty \le \|u(.,t)\|_\infty + \|U_k(t) - u_k(t)\|_\infty \quad \text{for } t \in (0, t(k)),$$

which implies that

$$\|U_k(t)\|_\infty \le \zeta + \frac{\beta-\zeta}{2} = \frac{\beta+\zeta}{2} < 1, \quad \text{for } t \in (0, t(k)). \tag{3.4}$$

Let $E_k(t) = U_k(t) - u_k(t)$ be the error of discretization. Using Taylor's expansion, we get

$$u_t(x_0,t) = \alpha(u(x_0,t))\frac{2u(x_1,t) - 2u(x_0,t)}{k^2} + f(x_0)\alpha(u(x_0,t))(1-u(x_0,t))^{-p} -$$
$$\alpha(u(x_0,t))\left(\frac{k}{3}u_{xxx}(\tilde{x}_0,t) + \frac{k^2}{12}u_{xxxx}(\tilde{x}_0,t)\right),$$
$$u_t(x_i,t) = \alpha(u(x_i,t))\frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{k^2} +$$
$$f(x_i)\alpha(u(x_i,t))(1-u(x_i,t))^{-p} - \alpha(u(x_i,t))\frac{k^2}{12}u_{xxxx}(\tilde{x}_i,t), \quad 1 \le i \le I-1,$$
$$u_t(x_I,t) = \alpha(u(x_I,t))\frac{2u(x_{I-1},t) - 2u(x_I,t)}{k^2} - \frac{2}{k}\alpha(u(x_I,t))g(u(x_I,t)) +$$
$$f(x_I)\alpha(u(x_I,t))(1-u(x_I,t))^{-p} + \alpha(u(x_I,t))\left(\frac{k}{3}u_{xxx}(\tilde{x}_I,t) - \frac{k^2}{12}u_{xxxx}(\tilde{x}_I,t)\right).$$

Which implies that

$$\frac{dE_0(t)}{dt} = \alpha(u(x_0,t))\delta^2 E_0(t)+$$
$$[pf_0\alpha(u(x_0,t))(1-\beta_0(t))^{-p-1} + f_0\alpha'(\eta_0(t))(1-U_0(t))^{-p} + \alpha'(\eta_0(t))\delta^2 U_0(t)]E_0(t)+$$
$$(f_0 - f(x_0))\alpha(u(x_0,t))(1-u(x_0,t))^{-p} + \alpha(u(x_0,t)\frac{k}{3}u_{xxx}(\tilde{x}_0,t) + \alpha(u(x_0,t)\frac{k^2}{12}u_{xxxx}(\tilde{x}_0,t),$$

$$\frac{dE_i(t)}{dt} = \alpha(u(x_i,t))\delta^2 E_i(t)+$$
$$[pf_i\alpha(u(x_i,t))(1-\beta_i(t))^{-p-1} + f_i\alpha'(\eta_i(t))(1-U_i(t))^{-p} + \alpha'(\eta_i(t))\delta^2 U_i(t)]E_i(t)$$
$$+(f_i - f(x_i))\alpha(u(x_i,t))(1-u(x_i,t))^{-p} + \alpha(u(x_i,t)\frac{k^2}{12}u_{xxxx}(\tilde{x}_i,t),$$

$1 \le i \le I-1$,

$$\frac{dE_I(t)}{dt} = \gamma(u(x_I,t))\delta^2 E_I(t)+$$
$$[pf_I\alpha(u(x_i,t))(1-\beta_I(t))^{-p-1} + f_I\alpha'(\eta_I(t))(1-U_I(t))^{-p} -$$
$$\frac{2\alpha(u(x_I,t))}{k}g'(\vartheta_I(t)) - \frac{2\alpha'(\eta_I(t))}{k}g(U_I(t)) + \alpha'(\eta_I(t))\delta^2 U_I(t)]E_I(t)$$
$$+(f_I - f(x_I))\alpha(u(x_I,t))(1-u(x_I,t))^{-p} - \alpha(u(x_I,t)\frac{k}{3}u_{xxx}(\tilde{x}_I,t)$$
$$+\alpha(u(x_I,t)\frac{k^2}{12}u_{xxxx}(\tilde{x}_I,t).$$

Using (3.2), (3.4) and taking into account that f is a continuous function, there exist M and N nonnegative constants such that

$$\frac{dE_0(t)}{dt} - \delta^2 E_0(t) \le M|E_0(t)| + Nk,$$
$$\frac{dE_i(t)}{dt} - \delta^2 E_i(t) \le M|E_i(t)| + Nk^2, \quad 1 \le i \le I-1,$$
$$\frac{dE_I(t)}{dt} - \delta^2 E_I(t) \le \frac{M}{k}|E_I(t)| + Nk.$$

Let $H \in C^{4,1}([0,1],[0,T])$ be such that

$$H(x,t) = e^{(M+1)t+d(-x^2+1)}(||\varphi_k - u_k(0)||_\infty + Nk)$$

where d is a nonnegative constant. A simple calculation give

$$\frac{dH_0(t)}{dt} - \delta^2 H_0(t) > M|H_0(t)| + Nk,$$
$$\frac{dH_i(t)}{dt} - \delta^2 H_i(t) > M|H_i(t)| + Nk^2, \quad 1 \le i \le I-1,$$
$$\frac{dH_I(t)}{dt} - \delta^2 H_I(t) > \frac{M}{k}|H_I(t)| + Nk,$$
$$H_i(0) > E_i(0), \quad 0 \le i \le I.$$

From Lemma 2.2, we obtain

$$H_i(t) > E_i(t), t \in (0, t(k)), \quad 0 \le i \le I.$$

By analogy, we also prove that

$$H_i(t) > -E_i(t), t \in (0, t(k)), \quad 0 \le i \le I.$$

Hence we have

$$H_i(t) > |E_i(t)|, t \in (0, t(k)), \quad 0 \le i \le I.$$

We deduce that

$$\|U_k(t) - u_k(t)\|_\infty \le (\|\varphi_k - u_k(0)\|_\infty + Nk)e^{(M+1)t+d}, t \in (0, t(k)).$$

Next we prove that $t(k) = T$. Suppose that $t(k) < T$. From (3.3), we obtain

$$\frac{\beta - \zeta}{2} \le \|U_k(t(k)) - u_k(t(k))\|_\infty \le (\|\varphi_k - u_k(0)\|_\infty + Nk)e^{(M+1)T+d}.$$

Since $(\|\varphi_k - u_k(0)\|_\infty + Nk)e^{(M+1)T+d} \longrightarrow 0$ as $k \longrightarrow 0$, we deduce that $\frac{\beta - \zeta}{2} \le 0$, which is impossible. Hence we have $t(k) = T$, and the proof is complete. □

4. Semidiscrete Quenching Time and Convergence

In this section, we show that under some assumptions, the solution U_k of (1.10)–(1.13) quenches in a finite time and estimate its semidiscrete quenching time.

Theorem 4.1. *Let U_k be a solution of (1.10)–(1.13), and assume that there exist a nonnegative constant $\tau \in (0, 1]$ such that the initial data at (1.13) satisfies*

$$\alpha(\varphi_i)\delta^2\varphi_i + \alpha(\varphi_i)f_i(1 - \varphi_i)^{-p} \ge \tau(1 - \varphi_i)^{-p}, \quad 0 \le i \le I - 1, \tag{4.1}$$

$$\alpha(\varphi_I)\delta^2\varphi_I + \alpha(\varphi_I)f_I(1 - \varphi_I)^{-p} - \frac{2\alpha(\varphi_I)}{k}g(\varphi_I) \ge \tau(1 - \varphi_I)^{-p}, \tag{4.2}$$

Then, the solution U_h quenches in a finite time T_q^k and we have the following estimate

$$T_q^k \le \frac{(1 - \|\varphi_k\|_\infty)^{p+1}}{\tau(p + 1)}.$$

Proof. Let $[0, T_q^k)$ be the maximal time interval on which $\|U_k\|_\infty < 1$. We consider the function $G_k(t)$ defined as follows

$$G_i(t) = \frac{dU_i(t)}{dt} - \tau(1 - U_i(t))^{-p}, \quad 0 \le i \le I, \quad [0, T_q^k). \tag{4.3}$$

By a straightforward computation we get

$$\frac{dG_i(t)}{dt} - \alpha(U_i(t))\delta^2 G_i(t) = \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \alpha(U_i(t))\delta^2 U_i(t)\right) +$$
$$\alpha'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t) - \tau p(1 - U_i(t))^{-p-1}\frac{dU_i(t)}{dt} +$$
$$\tau\alpha(U_i(t))\delta^2(1 - U_i(t))^{-p}, \quad 0 \le i \le I.$$

Since $\tau\delta^2(1 - U_i(t))^{-p} \ge p\tau(1 - U_i(t))^{-p-1}\delta^2 U_i(t)$, $0 \le i \le I$, we get

$$\frac{dG_i(t)}{dt} - \alpha(U_i(t))\delta^2 G_i(t) \ge \frac{d}{dt}\left(\frac{dU_i(t)}{dt} - \alpha(U_i(t))\delta^2 U_i(t)\right)$$
$$+\alpha'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t) - \tau p(1 - U_i(t))^{-p-1}\frac{dU_i(t)}{dt}$$

$$+\tau p\alpha(U_i(t))(1-U_i(t))^{-p-1}\delta^2 U_i(t),$$

$$\frac{dG_i(t)}{dt}-\alpha(U_i(t))\delta^2 G_i(t)\geq\frac{d}{dt}\left(\frac{dU_i(t)}{dt}-\alpha(U_i(t))\delta^2 U_i(t)\right)$$
$$-\tau p(1-U_i(t))^{-p-1}(\frac{dU_i(t)}{dt}-\alpha(U_i(t))\delta^2 U_i(t))$$
$$+\alpha'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t),\quad 0\leq i\leq I.$$

For $i\in\{0,...,I-1\}$, we get

$$\frac{dG_i(t)}{dt}-\alpha(U_i(t))\delta^2 G_i(t)\geq f_i\frac{d}{dt}\left[\alpha(U_i(t))(1-U_i(t))^{-p}\right]-$$
$$f_i\tau p(1-U_i(t))^{-p-1}\alpha(U_i(t))(1-U_i(t))^{-p}+\alpha'(U_i(t))\frac{dU_i(t)}{dt}\delta^2 U_i(t).$$

For $i=I$ we get

$$\frac{dG_I(t)}{dt}-\alpha(U_I(t))\delta^2 G_I(t)\geq\frac{d}{dt}\left(f_I\alpha(U_I(t))(1-U_I(t))^{-p}-\frac{2\alpha(U_I(t))}{k}g(U_I(t))\right)$$
$$-\tau p(1-U_I(t))^{-p-1}\left[f_I\alpha(U_I(t))(1-U_I(t))^{-p}-\frac{2\alpha(U_I(t))}{k}g(U_I(t))\right]$$
$$+\alpha'(U_I(t))\frac{dU_I(t)}{dt}\delta^2 U_I(t).$$

Which implies that

$$\frac{dG_i(t)}{dt}-\alpha(U_i(t))\delta^2 G_i(t)\geq pf_i\alpha(U_i(t))(1-U_i(t))^{-p-1}\left[\frac{dU_i(t)}{dt}-\tau(1-U_i(t))^{-p}\right]$$
$$+\alpha'(U_i(t))\frac{dU_i(t)}{dt}\left(\delta^2 U_i(t)+f_i(1-U_i(t))^{-p}\right),\quad 0\leq i\leq I-1,$$

$$\frac{dG_I(t)}{dt}-\alpha(U_I(t))\delta^2 G_I(t)\geq pf_I\alpha(U_I(t))(1-U_I(t))^{-p-1}\left(\frac{dU_I(t)}{dt}-\tau(1-U_I(t))^{-p}\right)$$
$$+\alpha'(U_I(t))\frac{dU_I(t)}{dt}\left(\delta^2 U_I(t)+f_I(1-U_I(t))^{-p}-\frac{2}{k}g(U_I(t))\right)+$$
$$\frac{2}{k}\alpha(U_I(t))\left(p\tau(1-U_I(t))^{-p-1}g(U_I(t))-g'(U_I(t))\frac{dU_I(t)}{dt}\right).$$

Finally, we get

$$\frac{dG_i(t)}{dt}-\alpha(U_i(t))\delta^2 G_i(t)\geq p\alpha(U_i(t))(1-U_i(t))^{-p-1}G_i(t),\quad 0\leq i\leq I.$$

From (4.1)–(4.2), we observe that $G_i(0)\geq 0$ for $0\leq i\leq I$. We deduce from Lemma 2.1 that $G_i(t)\geq 0$, $0\leq i\leq I$. Which implies that

$$dU_i(t)\geq\tau(1-U_i(t))^{-p}dt,\quad 0\leq i\leq I,\ t\in[0,T_q^k).$$

These inequalities can be rewritten as follows

$$(1-U_i(t))^p dU_i(t)\geq\tau dt,\quad 0\leq i\leq I,\ t\in[0,T_q^k).$$

Integrating the above inequalities over the interval (t, T_q^k), we get

$$T_q^k - t \leq \frac{(1-U_i(t))^{p+1}}{\tau(p+1)}, \quad 0 \leq i \leq I, \ t \in [0, T_q^k). \tag{4.4}$$

Taking $t = 0$ and $i = 0$, we obtain:

$$T_q^k \leq \frac{(1-\varphi_0)^{p+1}}{\tau(p+1)}.$$

Using the fact that $\|\varphi_k\|_\infty = \varphi_0$ thanks to the Lemma 2.3, we get:

$$T_q^k \leq \frac{(1-\|\varphi_k\|_\infty)^{p+1}}{\tau(p+1)}.$$

We have the desired result. □

Remark 4.1. By replacing t by t_0 and i by 0 in (4.4), we obtain

$$T_q^k - t_0 \leq \frac{(1-\|U_k(t_0)\|_\infty)^{p+1}}{\tau(p+1)}, \quad t_0 \in [0, T_q^k),$$

and

$$\|U_k(t_0)\|_\infty \leq 1 - Q_1(T_q^k - t_0)^{\frac{1}{p+1}},$$

where $Q_1 = (\tau(p+1))^{\frac{1}{p+1}}$.

The Remark 4.1 is crucial to prove the convergence of the semidiscrete quenching time.

Theorem 4.2. *Suppose that the solution u of problem (1.7)–(1.9) quenches in a finite time T_q such that $u \in C^{4,1}([0,1]\times[0,T_q))$ and the initial data at (1.13) satisfies*

$$\|\varphi_k - u_k(0)\|_\infty = o(1) \quad \text{as } k \longrightarrow 0. \tag{4.5}$$

Under the assumptions of Theorem 4.1, the solution U_k of (1.10)–(1.13) quenches in finite time T_q^k and we have

$$\lim_{k\to 0} T_q^k = T_q.$$

Proof. Set $0 < \varepsilon < \frac{T_q}{2}$. There exists $\eta = \beta - \zeta$ (with $0 < \zeta < \beta < 1$) such that

$$\frac{(1-\varrho)^{p+1}}{\tau(p+1)} \leq \frac{\varepsilon}{2}, \quad \varrho \in [1-\eta, 1). \tag{4.6}$$

Since $\lim_{t\to T_q^-} \|u(.,t)\|_\infty = 1$, there exists a time $T_1 < T_q$ and $|T_q - T_1| < \dfrac{\varepsilon}{2}$ such that $1-\frac{\eta}{2} \leq \|u(.,t)\|_\infty < 1$ for $t \in [T_1, T_q)$. From Theorem 3.1, the problem (1.10)–(1.13) has for each k, a unique solution U_k such that $\|U_k(t) - u_k(t)\|_\infty < \dfrac{\eta}{2}$ for $t \in [0, T_2]$ where $T_2 = \frac{T_1+T_q}{2}$. Using the triangle inequality, we get

$$\|U_k(t)\|_\infty \geq \|u(.,t)\|_\infty - \|U_k(t) - u_k(t)\|_\infty \geq 1 - \frac{\eta}{2} - \frac{\eta}{2} : \text{ for } : t \in [T_1, T_2].$$

which implies that

$$\|U_k(t)\|_\infty \geq 1 - \eta : \text{ for } : t \in [T_1, T_2].$$

From Theorem 4.1, U_k quenches in a finite time T_q^k. We deduce from Remark 4.1 and (4.6) that

$$|T_q^k - T_1| \leq \frac{(1 - \|U_k(T_1)\|_\infty)^{p+1}}{\tau(p+1)} \leq \frac{\varepsilon}{2},$$

which implies

$$|T_q^k - T_q| \leq |T_q^k - T_1| + |T_1 - T_q| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,$$

and the proof is complete. □

5. Numerical Tests

In this section, we present some numerical approximations of the quenching time of the problem (1.4)–(1.6) in the case where $\varphi_i = 0.7 - \frac{1}{2}(ik)^4$, $f_i = (1 + ik)^{-0.5}$, $\alpha(U_i^{(n)}) = \frac{(U_i^{(n)})^{(1-\beta)}}{\beta}$, $g(U_i^{(n)}) = (U_i^{(n)})^{-\varepsilon}, 0 \leq i \leq I$ where $0 < \beta \leq 1$ and $\varepsilon > 0$. Firstly, we consider the following explicit scheme

$$U_0^{(n+1)} = a_0^{(n)} U_0^{(n)} + 2b_0^{(n)} U_1^{(n)} + c_0^{(n)} (1 - U_0^{(n)})^{-p},$$

$$U_i^{(n+1)} = a_i^{(n)} U_i^{(n)} + b_i^{(n)} (U_{i-1}^{(n)} + U_{i+1}^{(n)}) + c_i^{(n)} (1 - U_i^{(n)})^{-p}, \qquad 1 \leq i \leq I-1,$$

$$U_I^{(n+1)} = a_I^{(n)} U_I^{(n)} + 2b_I^{(n)} U_{I-1}^{(n)} + c_I^{(n)} (1 - U_I^{(n)})^{-p} + d_I^{(n)} (U_I^{(n)})^{-\varepsilon},$$

$$U_i^{(0)} = \varphi_i, \qquad 0 \leq i \leq I,$$

where

$$a_i^{(n)} = 1 - 2\Delta t_n^e \frac{(U_i^{(n)})^{(1-\beta)}}{\beta k^2}, \qquad 0 \leq i \leq I, \quad n \geq 0,$$

$$b_i^{(n)} = \Delta t_n^e \frac{(U_i^{(n)})^{(1-\beta)}}{\beta k^2}, \qquad 0 \leq i \leq I, \quad n \geq 0,$$

$$c_i^{(n)} = f_i \Delta t_n^e \frac{(U_i^{(n)})^{(1-\beta)}}{\beta}, \qquad 0 \leq i \leq I, \quad n \geq 0,$$

$$d_I^{(n)} = -2\Delta t_n^e \frac{(U_I^{(n)})^{(1-\beta)}}{\beta k}, \qquad n \geq 0,$$

$$\Delta t_n^e = \min \left\{ \frac{k^2}{2}, k^2 (1 - \|U_k^{(n)}\|_\infty)^{p+1} \right\}.$$

We also consider the implicit scheme

$$U_0^{(n)} = \eta_0^{(n)} U_0^{(n+1)} + 2\vartheta_0^{(n)} U_1^{(n+1)} + \varsigma_0^{(n)} (1 - U_0^{(n)})^{-p},$$

$$U_i^{(n)} = \eta_i^{(n)} U_i^{(n+1)} + \vartheta_i^{(n)} (U_{i-1}^{(n+1)} + U_{i+1}^{(n+1)}) + \varsigma_i^{(n)} (1 - U_i^{(n)})^{-p}, \qquad 1 \leq i \leq I-1,$$

$$U_I^{(n)} = \eta_I^{(n)} U_I^{(n+1)} + 2\vartheta_I^{(n)} U_{I-1}^{(n+1)} + \varsigma_I^{(n)} (1 - U_I^{(n)})^{-p} + \sigma_I^{(n)} (U_I^{(n)})^{-\varepsilon},$$

$$U_i^{(0)} = \varphi_i, \qquad 0 \leq i \leq I,$$

where

$$\eta_i^{(n)} = 1 + 2\Delta t_n \frac{(U_i^{(n)})^{(1-\beta)}}{\beta k^2}, \qquad 0 \leq i \leq I, \quad n \geq 0,$$

$$\vartheta_i^{(n)} = -\Delta t_n \frac{(U_i^{(n)})^{(1-\beta)}}{\beta k^2}, \qquad 0 \leq i \leq I, \quad n \geq 0,$$

$$\varsigma_i^{(n)} = -f_i \Delta t_n \frac{(U_i^{(n)})^{(1-\beta)}}{\beta}, \qquad 0 \leq i \leq I, \quad n \geq 0,$$

$$\sigma_I^{(n)} = 2\Delta t_n \frac{(U_I^{(n)})^{(1-\beta)}}{\beta k}, \qquad n \geq 0,$$

$$\Delta t_n = k^2 (1 - \|U_k^{(n)}\|_\infty)^{p+1}.$$

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations and the orders of the approximations corresponding to meshes $16, 32, 64, 128, 256, 512$. The numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

The order s of the method is computed from

$$s = \frac{log((T_{4k} - T_{2k})/(T_{2k} - T_k))}{log(2)}.$$

Table 1. Numerical quenching times obtained with the explicit Euler method $p = 2$, $\beta = 0.25$ and $\varepsilon = -log(2)/log(0.2)$

I	T^n	n	s
16	0.002870572	611	-
32	0.002849315	2346	-
64	0.002844171	8950	2.00
128	0.002842906	34012	2.00
256	0.002842594	128838	2.00
512	0.002842517	486376	2.00

Table 2. Numerical quenching times obtained with the implicit Euler method $p = 2$, $\beta = 0.25$ and $\varepsilon = -log(2)/log(0.2)$

I	T^n	n	s
16	0.002874225	611	-
32	0.002850205	2346	-
64	0.002844392	8950	2.00
128	0.002842961	34012	2.00
256	0.002842608	128838	2.00
512	0.002842520	486376	2.00

Table 3. Numerical quenching times obtained with the explicit Euler method $p = 2$, $\beta = 0.5$ and $\varepsilon = 0.5$

I	T^n	n	s
16	0.005343498	1214	-
32	0.005328986	4645	-
64	0.005325609	17697	2.00
128	0.005324800	67195	2.00
256	0.005324605	254324	2.00
512	0.005324557	959235	2.00

Table 4. Numerical quenching times obtained with the implicit Euler method $p = 2$, $\beta = 0.5$ and $\varepsilon = 0.5$

I	T^n	n	s
16	0.005347072	1214	-
32	0.005329857	4645	-
64	0.005325826	17697	2.00
128	0.005324854	67195	2.00
256	0.005324618	254324	2.00
512	0.005324561	959235	2.00

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 64$ and $(p; \beta; \varepsilon) = (2; 0.5; 0.5)$.

Figures (1), (2) show that the numerical solution quenches.

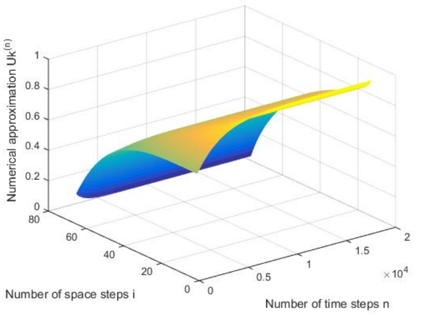

FIGURE 1. Evolution of the numerical solution (explicit scheme).

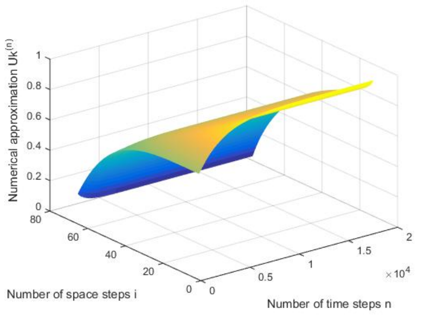

FIGURE 2. Evolution of the numerical solution (implicit scheme).

In figures (3), (4), we can appreciate that the numerical solution quenches at the first node and decreases with respect to the spatial variable.

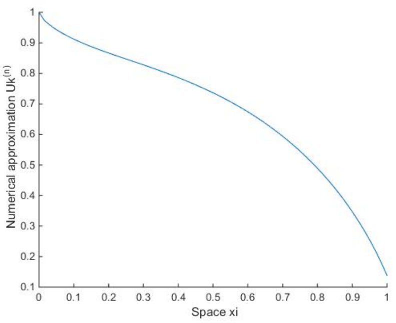

FIGURE 3. The profile of the approximation of u(x,T) where, T is the quenching time (explicit scheme).

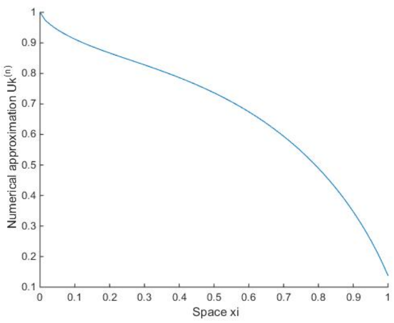

FIGURE 4. The profile of the approximation of u(x,T) where, T is the quenching time (implicit scheme).

Figures (5), (6) show us that the numerical solution quenches at finite time $T^n \approx 5.4 \times 10^{-3}$ and is increasing with respect to the time variable.

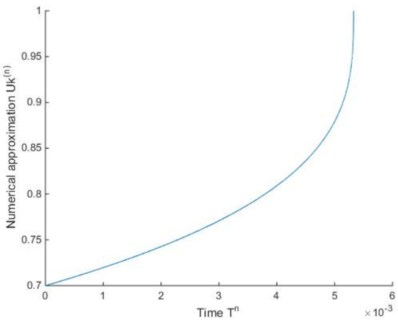

FIGURE 5. The profile of the approximation of $\|U_k^{(n)}\|_\infty$ (explicit scheme).

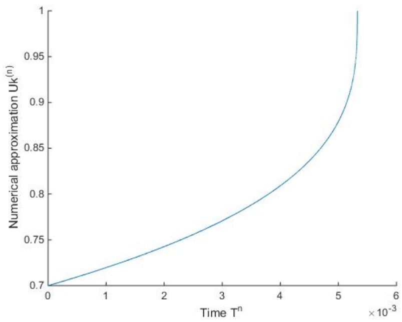

FIGURE 6. The profile of the approximation of $\|U_k^{(n)}\|_\infty$ (implicit scheme).

CONCLUSION

At the end of our work, we have been able to confirm the finite-time quenching behaviour of the numerical solution to problem (1.1)–(1.3). By constructing and analysing a finite-difference approximation of the continuous formulation, we established both the reliability and the convergence of our numerical approach. Moreover, the estimate of the numerical quenching time, together with the consistency of the results obtained through our tests, reinforces the validity of the method and highlights its capacity to capture the essential dynamics of the problem. These findings open the way for further numerical investigations and potential extensions of the model.

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