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ON η-RICCI SOLITONS ON HOMOTHETIC KENMOTSU MANIFOLDS WITH RESPECT TO A GENERAL CONNECTION

MURAT ALTUNBAŞ

Abstract. We study a connection which generalizes the quarter-symmetric metric, the Schouten-Van Kampen, the Tanaka-Webster and the generalized Tanaka-Webster connections. With the help of this connection, we characterize η-Ricci solitons on homothetic Kenmotsu manifolds. In the final part, we give an example to verify some obtained results.

1. Introduction

Let (M, g, η, ξ, ϕ) be an almost contact metric manifold. In [1], [2], [3], a general connection $\bar{\nabla}$ is defined by

$$\bar{\nabla}_U V = \nabla_U V + c_1[(\nabla_U \eta)(V)\xi - \eta(V)\nabla_U \xi] + c_2\eta(U)\phi V, \qquad (1.1)$$

for all vector fields $U, V \in \chi(M)$ and $c_1, c_2 \in \mathbb{R}$, where $\chi(M)$ is the set of all differentiable vector fields on M and ∇ is the Levi-Civita connection of the metric g. The connection $\bar{\nabla}$ is a general connection, since

(i) if $c_1 = 0, c_2 = -1$, we obtain the quarter-symmetric metric connection [6],

(ii) if $c_1 = 1, c_2 = 0$, we obtain the Schouten-Van Kampen connection [13],

(iii) if $c_1 = 1, c_2 = -1$, we obtain the Tanaka-Webster connection [14],

(iv) if $c_1 = 1, c_2 = 1$, we obtain the generalized Tanaka-Webster connection [15].

In [9], Oubina introduced the concept of (α, β)-trans-Sasakian manifolds which contain both the class of Sasakian and cosymplectic structures and are related to the locally conformal Kähler manifolds. The trans-Sasakian manifolds of type $(0, 0), (\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, α-Sasakian and β-Kenmotsu manifolds, respectively. Here, α and β denote some scalar functions. In particular, if $\alpha = 0, \beta = 1$, the trans-Sasakian manifolds are said to be Kenmotsu manifolds. Moreover, if $\alpha = 0$ and β is a constant function, the trans-Sasakian manifolds are called homothetic Kenmotsu manifolds [7].

The notion of η-Ricci soliton was introduced by Cho and Kimura in [5]. In this paper, the authors gave a classification of real hypersurfaces in nonflat complex space

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forms admitting η-Ricci solitons. An η-Ricci soliton (g, X, λ, μ) on a Riemannian manifold (M, g) is defined by

$$(\mathfrak{L}_X g)(U, V) + 2S(U, V) + 2\lambda g(U, V) + 2\mu\eta(U)\eta(V) = 0, \tag{1.2}$$

where $\lambda, \mu \in \mathbb{R}$. An η-Ricci soliton is called steady if $\lambda = 0$, shrinking if $\lambda < 0$ and expanding if $\lambda > 0$. When $\mu = 0$, we obtain the well-known Ricci soliton. We may refer to [4],[8],[12], [19] for some recent works on η-Ricci solitons.

This paper is organized as follows: In Section 2, we recall some basic results about homothetic Kenmotsu manifolds. In Section 3, we obtain the Riemannian curvature tensor, Ricci tensor and the scalar curvature of a homothetic Kenmotsu manifold with respect to the general connection $\bar{\nabla}$. In Section 4, we give some characterizations of η-Ricci solitons admitting the general connection $\bar{\nabla}$. Finally, we construct an example of a 3-dimensional homothetic Kenmotsu manifold admitting the general connection $\bar{\nabla}$ in order to confirm some results.

2. Homothetic Kenmotsu manifolds

An $n = (2k+1)$-dimensional smooth manifold M is called an almost contact metric manifold if it admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Riemannian metric g which fulfill

$$\phi^2(U) = -U + \eta(U)\xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta(\phi U) = 0, \tag{2.1}$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \ g(\phi U, V) = -g(U, \phi V),$$
$$g(U, \xi) = \eta(U), \ \forall U, V \in \chi(M).$$

An almost contact metric manifold (M, g, η, ξ, ϕ) is called β-Kenmotsu if

$$(\nabla_U \phi)(V) = \beta[g(\phi U, V)\xi - \eta(V)\phi U]. \tag{2.2}$$

From (2.2), we have

$$\nabla_U \xi = \beta[U - \eta(U)\xi], \tag{2.3}$$

where β is a scalar function on M and ∇ is the Levi-Civita connection of g. If $\beta = 1$ then the β-Kenmotsu manifold is called Kenmotsu manifold and if β is constant then the β-Kenmotsu manifold is called homothetic Kenmotsu manifold [16].

In a homothetic Kenmotsu manifold, the following relations are satisfied:

$$(\nabla_U \eta)V = \beta[g(U, V) - \eta(U)\eta(V)], \tag{2.4}$$

$$R(U, V)\xi = -\beta^2[\eta(V)U - \eta(U)V],$$

$$S(U, \xi) = -(n-1)\beta^2\eta(U),$$

$$R(\xi, U)V = \beta^2[\eta(V)U - g(U, V)\xi],$$

$$Q\xi = -(n-1)\beta^2\xi,$$

where R, S and Q denote the Riemannian curvature tensor, Ricci tensor and the Ricci operator, respectively.

3. Curvatures of homothetic Kenmotsu manifolds with a general connection

Using (2.3) and (2.4) in (1.1), the general connection $\bar{\nabla}$ is expressed as

$$\bar{\nabla}_U V = \nabla_U V + c_1\beta[g(U,V)\xi - \eta(V)U] + c_2\eta(U)\phi V. \tag{3.1}$$

Replacing V by ξ in (3.1) and using (2.1), (2.3), we get

$$\bar{\nabla}_U \xi = (1-c_1)\beta[U - \eta(U)\xi]. \tag{3.2}$$

From (2.1)-(2.4) and (3.1), we obtain

$$U(\eta(V)) = \eta(\nabla_U V) + \beta g(U,V) - \beta\eta(U)\eta(V),$$

$$\bar{\nabla}_U(\phi V) = \nabla_U(\phi V) + c_1\beta g(U,\phi V)\xi - c_2\eta(U)V + c_2\eta(U)\eta(V)\xi,$$

$$(\bar{\nabla}_U g)(V,W) = 0. \tag{3.3}$$

Now, we can compute the curvature tensor $\bar{R}$ of the connection $\bar{\nabla}$ using the following formula:

$$\bar{R}(U,V)W = \bar{\nabla}_U\bar{\nabla}_V W - \bar{\nabla}_V\bar{\nabla}_U W - \bar{\nabla}_{[U,V]}W. \tag{3.4}$$

Taking into account (3.1)-(3.3), we deduce

$$\begin{aligned} \bar{\nabla}_{[U,V]}W &= \nabla_{[U,V]}W + c_2\eta(\nabla_U V)\phi W - c_2\eta(\nabla_V U)\phi W \\ &\quad + c_1\beta[g(\nabla_U V,W)\xi - g(\nabla_V U,W)\xi - \eta(W)\nabla_U V + \eta(W)\nabla_V U], \end{aligned} \tag{3.5}$$

$$\begin{aligned} \bar{\nabla}_U\bar{\nabla}_V W &= \nabla_U\nabla_V W + c_1\beta g(U,\nabla_V W)\xi - c_1\beta\eta(\nabla_V W)U + c_2\eta(U)\phi\nabla_V W \\ &\quad + c_1\beta[g(\nabla_U V,W) + c_1\beta g(U,V)\eta(W) - c_1\beta\eta(V)g(U,W) + c_2\eta(U)g(\phi V,W) \\ &\quad + g(\nabla_U W,V) + c_1\beta g(U,W)\eta(V) - c_1\beta\eta(W)g(U,V) + c_2\eta(U)g(\phi W,V)]\xi \\ &\quad + c_1\beta g(V,W)(1-c_1)\beta(U - \eta(U)\xi) - c_1\beta[\eta(\nabla_U W) + \beta g(U,W) \\ &\quad - \beta\eta(U)\eta(W)]V - c_1\beta\eta(W)[\nabla_U V + c_1\beta(g(U,V)\xi - \eta(V)U) + c_2\eta(U)\phi V] \\ &\quad + c_2[\eta(\nabla_U V) + \beta g(U,V) - \beta\eta(U)\eta(V)]\phi W + c_2\eta(V)[\beta(-g(U,\phi W)) \\ &\quad + \phi\nabla_U W + c_1\beta g(U,\phi W)\xi - c_2\eta(U)W + c_2\eta(U)\eta(W)\xi]. \end{aligned} \tag{3.6}$$

If we interchange U and V in (3.6), we get

$$\begin{aligned} \bar{\nabla}_V\bar{\nabla}_U W &= \nabla_V\nabla_U W + c_1\beta g(V,\nabla_U W)\xi - c_1\beta\eta(\nabla_U W)V + c_2\eta(V)\phi\nabla_U W \\ &\quad + c_1\beta[g(\nabla_V U,W) + c_1\beta g(V,U)\eta(W) - c_1\beta\eta(U)g(V,W) + c_2\eta(V)g(\phi U,W) \\ &\quad + g(\nabla_V W,U) + c_1\beta g(V,W)\eta(U) - c_1\beta\eta(W)g(V,U) + c_2\eta(V)g(\phi W,U)]\xi \\ &\quad + c_1\beta g(U,W)(1-c_1)\beta(V - \eta(V)\xi) - c_1\beta[\eta(\nabla_V W) + \beta g(V,W) \\ &\quad - \beta\eta(V)\eta(W)]U - c_1\beta\eta(W)[\nabla_V U + c_1\beta(g(V,U)\xi - \eta(U)V) + c_2\eta(V)\phi U] \\ &\quad + c_2[\eta(\nabla_V U) + \beta g(V,U) - \beta\eta(V)\eta(U)]\phi W + c_2\eta(U)[\beta(-g(V,\phi W)) \\ &\quad + \phi\nabla_V W + c_1\beta g(V,\phi W)\xi - c_2\eta(V)W + c_2\eta(V)\eta(W)\xi]. \end{aligned} \tag{3.7}$$

Putting (3.5), (3.6) and (3.7) in (3.4), we obtain

$$\begin{aligned}\bar{R}(U,V)W &= R(U,V)W + (c_1c_2 - c_2)\beta[\eta(V)g(U,\phi W)\xi - \eta(U)g(V,\phi W)\xi] \\ &+(c_1c_2 - c_2)\beta[\eta(V)\eta(W)\phi U - \eta(U)\eta(W)\phi V] \\ &+c_1(1-c_1)\beta^2[g(U,W)\eta(V)\xi - g(V,W)\eta(U)\xi] \\ &+c_1(2-c_1)\beta^2[g(V,W)U - g(U,W)V] \\ &+c_1(1-c_1)\beta^2[\eta(U)V - \eta(V)U]\eta(W).\end{aligned} \tag{3.8}$$

where R denotes the Riemannian curvature tensor of ∇. Therefore, we can state the following result.

Theorem 3.1. *Let M be an n-dimensional homothetic Kenmotsu manifold admitting the general connection $\bar{\nabla}$. Then the curvature tensor $\bar{R}$ of $\bar{\nabla}$ is given by (3.8).*

Taking contraction in (3.8), we obtain the Ricci tensor $\bar{S}$ of $\bar{\nabla}$ as

$$\begin{aligned}\bar{S}(V,W) &= S(V,W) + c_2(1-c_1)\beta g(V,\phi W) + (2-n)c_1(1-c_1)\beta^2\eta(V)\eta(W) \\ &+\beta^2[2nc_1 - nc_1^2 - 3c_1 + 2c_1^2]g(V,W),\end{aligned} \tag{3.9}$$

where S denotes the Ricci tensor of ∇. This gives the Ricci operator $\bar{Q}$ as

$$\begin{aligned}\bar{Q}V &= QV - c_2(1-c_1)\beta\phi V + (2-n)c_1(1-c_1)\beta^2\eta(V)\xi \\ &+\beta^2[2nc_1 - nc_1^2 - 3c_1 + 2c_1^2]V,\end{aligned} \tag{3.10}$$

where Q denotes the Ricci operator of ∇. Similarly, taking the contraction of (3.9), the scalar curvature $\bar{r}$ of $\bar{\nabla}$ is obtained by

$$\bar{r} = r + (2-n)c_1(1-c_1)\beta^2 + \beta^2 n[2nc_1 - nc_1^2 - 3c_1 + 2c_1^2], \tag{3.11}$$

where r denotes the scalar curvature of ∇. Thus, we can summarize our results in the following theorem.

Theorem 3.2. *Let M be an n-dimensional homothetic Kenmotsu manifold admitting the general connection $\bar{\nabla}$. Then*
(i) The Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is given by (3.9).
(ii) The Ricci operator $\bar{Q}$ of $\bar{\nabla}$ is given by (3.10).
(iii) The scalar curvature $\bar{r}$ of $\bar{\nabla}$ is given by (3.11).

On the other hand, using (3.9), we find

$$\bar{S}(V,\xi) = \beta^2\eta(V)[(n-1)(c_1-1)]. \tag{3.12}$$

From (3.8), we obtain

$$\begin{aligned}\bar{R}(\xi,V)W &= (1-c_1)\beta^2\eta(W)V - \beta^2(1-c_1)g(V,W)\xi \\ &-c_2(c_1-1)\beta[g(V,\phi W)\xi + \eta(W)\phi V] \\ &+c_1(1-c_1)\beta^2\eta(V)\eta(W)\xi,\end{aligned}$$

$$\bar{R}(V,W)\xi = (1-c_1)\beta^2\eta(V)W + c_1\beta^2\eta(W)V + c_2(c_1-1)\beta[\eta(W)\phi V - \eta(V)\phi W].$$

4. η-Ricci solitons on homothetic Kenmotsu manifolds admitting a general connection

In this section we characterize the η-Ricci solitons on homothetic Kenmotsu manifolds with respect to the general connection $\bar{\nabla}$. For the general connection $\bar{\nabla}$, we define

$$(\bar{\mathfrak{L}}_{\xi}g)(U,V) = g(\bar{\nabla}_U\xi, V) + g(U, \bar{\nabla}_V\xi). \tag{4.1}$$

Then, from (1.2), we have

$$(\bar{\mathfrak{L}}_{\xi}g)(U,V) + 2\bar{S}(U,V) + 2\lambda g(U,V) + 2\mu\eta(U)\eta(V) = 0. \tag{4.2}$$

Using (3.1), (3.2), the relation (4.2) becomes

$$(\bar{\mathfrak{L}}_{\xi}g)(U,V) = 2\beta(1-c_1)g(U,V) - 2\beta(1-c_1)\eta(U)\eta(V). \tag{4.3}$$

Taking into account (4.3), equation (4.2) reduces to

$$\bar{S}(U,V) = [\beta(1-c_1) - \mu]\eta(U)\eta(V) - [\beta(1-c_1) + \lambda]g(U,V). \tag{4.4}$$

Putting $U = V = \xi$ in (4.4), we obtain

$$\lambda + \mu = \beta^2(n-1)(1-c_1). \tag{4.5}$$

So, we have

Theorem 4.1. *If (M, g, η, ξ, ϕ) is a homothetic Kenmotsu manifold, (g, ξ, λ, μ) is an η-Ricci soliton with respect to the quarter-symmetric metric connection, then the η-Ricci soliton is shrinking if $\beta^2(n-1) < \mu$, steady if $\beta^2(n-1) = \mu$, expanding if $\beta^2(n-1) > \mu$.*

Theorem 4.2. *If (M, g, η, ξ, ϕ) is a homothetic Kenmotsu manifold, (g, ξ, λ, μ) is an η-Ricci soliton with respect to the Schouten-Van Kampen, Tanaka-Webster or generalized Tanaka-Webster connection, then the η-Ricci soliton is shrinking if $0 < \mu$, steady if $\mu = 0$, expanding if $\mu < 0$.*

Now, we define a T-like curvature tensor as in [18]. The importance of the T-curvature tensor comes from its generalizing some well-known curvature tensors such as conharmonic [17], M-projective [20], and W_i-curvatures [11].

Definition 4.1. A homothetic Kenmotsu manifold is called T-like flat with respect to the general connection $\bar{\nabla}$ if

$$\bar{T}(U,V)W = 0,$$

where $\bar{T}$ is a T-like curvature tensor defined by

$$\begin{aligned} \bar{T}(U,V)W \;\; = \;\; & a_0\bar{R}(U,V)W + \\ & a_1\bar{S}(V,W)U + a_2\bar{S}(U,W)V + a_3\bar{S}(U,V)W \\ & + a_4 g(V,W)\bar{Q}U + a_5 g(U,W)\bar{Q}V + a_6 g(U,V)\bar{Q}W \\ & + a_7\bar{r}(g(V,W)U - g(U,W)V), \end{aligned}$$

where $a_0, \cdots, a_7$ are real constants.

Assume that

$$a_0 + a_2 + a_3 + na_4 + a_5 + a_6 \neq 0.$$

$$\bar{S}(U, X) = -\frac{[a_1 + (n-1)a_7]\bar{r}}{a_0 + a_2 + a_3 + na_4 + a_5 + a_6} g(U, X). \tag{4.6}$$

Using (4.4) and (4.6), we obtain

$$\begin{aligned} & [\beta(1-c_1) - \mu]\eta(U)\eta(X) - [\beta(1-c_1) + \lambda]g(U, X) \\ = & -\frac{[a_1 + (n-1)a_7]\bar{r}}{a_0 + a_2 + a_3 + na_4 + a_5 + a_6} g(U, X). \end{aligned} \tag{4.7}$$

Taking $U = X = \xi$ in (4.7), we find

$$\mu + \lambda = \frac{a_1 + (n-1)a_7}{a_0 + a_2 + a_3 + na_4 + a_5 + a_6}[r + (2-n)c_1(1-c_1)\beta^2 + \beta^2 n(2nc_1 - nc_1^2 - 3c_1 + 2c_1^2)].$$

Thus, we can state the following theorem.

Theorem 4.3. *(i) If (g, ξ, λ, μ) is an η-Ricci soliton on a T-like flat homothetic Kenmotsu manifold with respect to the quarter-symmetric metric connection, then the relation between μ and λ is given by*

$$\mu + \lambda = \frac{[a_1 + (n-1)a_7]r}{a_0 + a_2 + a_3 + na_4 + a_5 + a_6}.$$

(ii) If (g, ξ, λ, μ) is an η-Ricci soliton on a T-like flat homothetic Kenmotsu manifold with respect to the Schouten-Van Kampen, Tanaka-Webster or generalized Tanaka-Webster connection, then the relation between μ and λ is given by

$$\mu + \lambda = \frac{a_1 + (n-1)a_7}{a_0 + a_2 + a_3 + na_4 + a_5 + a_6}[r + \beta^2 n(n-1)].$$

Now, consider the potential vector field X as a pointwise collinear with the vector field ξ, i.e., $X = \gamma\xi$, where γ is a function on M. From (4.2), we have

$$\begin{aligned} 0 = & (U\gamma)\eta(V) + (V\gamma)\eta(U) + 2\gamma\beta(1-c_1)[g(U, V) - \eta(U)\eta(V)] \\ & + 2\bar{S}(U, V) + 2\lambda g(U, V) + 2\mu\eta(U)\eta(V). \end{aligned} \tag{4.8}$$

Taking $V = \xi$ in (4.8) and having in mind (3.12), we find

$$(U\gamma) + (\xi\gamma)\eta(U) + 2\beta^2\eta(U)[(n-1)(c_1-1)] + 2\lambda\eta(U) + 2\mu\eta(U) = 0. \tag{4.9}$$

Taking $U = \xi$ in (4.9), we get

$$\xi\gamma + \beta^2(n-1)(c_1-1) + \lambda + \mu = 0. \tag{4.10}$$

Using (4.10) in (4.9), we obtain

$$(U\gamma) + [\beta^2(n-1)(c_1-1) + \lambda + \mu]\eta(U) = 0. \tag{4.11}$$

Covariant differentiation with respect to V gives us

$$[\beta^2(n-1)(c_1-1) + \lambda + \mu](\nabla_V\eta)(U) = 0. \tag{4.12}$$

Since $(\nabla_V\eta)(U) = \beta[g(U, V) - \eta(U)\eta(V)]$ is not identically zero for any vector fields U, V orthogonal to ξ, we deduce from (4.12) that

$$\lambda + \mu = -\beta^2(n-1)(c_1-1). \tag{4.13}$$

Putting (4.13) in (4.11) shows that γ is constant. Then from (4.8), we have

$$\bar{S}(U,V) = -[\gamma\beta(1-c_1)+\lambda]g(U,V) + [\gamma\beta(1-c_1)-\mu]\eta(U)\eta(V). \tag{4.14}$$

Hence $\bar{S}$ is of the form

$$\bar{S}(U,V) = Ag(U,V) + B\eta(U)\eta(V),$$

where

$$A = -[\gamma\beta(1-c_1)+\lambda], \qquad B = \gamma\beta(1-c_1)-\mu.$$

Therefore, we obtain the following theorem.

Theorem 4.4. *If (M, g, η, ξ, ϕ) is a homothetic Kenmotsu manifold, (g, ξ, λ, μ) is an η-Ricci soliton with respect to the general connection $\bar{\nabla}$ such that $X = \gamma\xi$ for some smooth function γ, then γ is constant and the Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is of the form*

$$\bar{S}(U,V) = Ag(U,V) + B\eta(U)\eta(V),$$

for some constants A and B.

5. Example

In this final section, we construct an example of a 3-dimensional homothetic Kenmotsu manifold admitting the general connection $\bar{\nabla}$ with the help of [10], in order to verify some of our results.

Let $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields $\{e_1, e_2, e_3\}$ as

$$e_1 = k_2 e^{-\beta z}\frac{\partial}{\partial x} + k_1 e^{-\beta z}\frac{\partial}{\partial y},\ e_2 = -k_1 e^{-\beta z}\frac{\partial}{\partial x} + k_2 e^{-\beta z}\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z},$$

where k_1, k_2, β are constants such that $k_1^2 + k_2^2 = 1$, $\beta \neq 0$. Let g be a Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \text{ for } i, j = 1, \cdots, 3.$$

Let η be the 1-form defined by $\eta(U) = g(U, e_3)$ for all vector fields on M and ϕ be the $(1,1)$-tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Also, let $\xi = e_3$. Then, we have

$$\phi^2 U = -U + \eta(U)e_3,\ \eta(e_3) = 1,\ g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V),\ \forall U, V \in \chi(M).$$

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we obtain

$$[e_1, e_2] = 0,\ [e_1, e_3] = \beta e_1,\ \ [e_2, e_3] = \beta e_2. \tag{5.1}$$

The Koszul formula

$$2g(\nabla_U V, W) = Ug(V,W) + Vg(W,U) - Wg(U,V) - g(U,[V,W]) + g(V,[W,U]) + g(W,[U,V])$$

gives us

$$\begin{aligned} &\nabla_{e_1} e_1 = -\beta e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = \beta e_1, \\ &\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -\beta e_3,\ \nabla_{e_2} e_3 = \beta e_2, \\ &\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0. \end{aligned} \tag{5.2}$$

Thus it can be seen that (M, g, η, ξ, ϕ) is a homothetic Kenmotsu manifold. Using (3.1), (3.4), (5.1), (5.2), we get

$$\bar{\nabla}_{e_1} e_1 = \beta(c_1 - 1)e_3,\ \bar{\nabla}_{e_1} e_2 = 0,\ \bar{\nabla}_{e_1} e_3 = \beta(1 - c_1)e_1,$$
$$\bar{\nabla}_{e_2} e_1 = 0,\ \bar{\nabla}_{e_2} e_2 = \beta(c_1 - 1)e_3,\ \bar{\nabla}_{e_2} e_3 = \beta(1 - c_1)e_2,$$
$$\bar{\nabla}_{e_3} e_1 = c_2 e_2,\ \bar{\nabla}_{e_3} e_2 = -c_2 e_1,\ \bar{\nabla}_{e_3} e_3 = 0,$$

$$\begin{aligned}
\bar{R}(e_2, e_1)e_1 &= -\beta^2(c_1 - 1)^2 e_2, \\
\bar{R}(e_3, e_1)e_1 &= \beta^2(c_1 - 1)e_3, \\
\bar{R}(e_1, e_2)e_2 &= -\beta^2(1 - c_1)^2 e_1, \\
\bar{R}(e_3, e_2)e_2 &= \beta^2(c_1 - 1)e_3, \\
\bar{R}(e_1, e_3)e_3 &= -c_2\beta(1 - c_1)e_2 - \beta^2(1 - c_1)e_1, \\
\bar{R}(e_2, e_3)e_3 &= c_2\beta(1 - c_1)e_1 - \beta^2(1 - c_1)e_2, \\
\bar{R}(e_1, e_3)e_2 &= -c_2\beta(c_1 - 1)e_3, \\
\bar{R}(e_1, e_2)e_3 &= 0,
\end{aligned}$$

$$\begin{aligned}
\bar{S}(e_1, e_1) &= -2\beta^2 - \beta^2 c_1^2 + 3\beta^2 c_1, \\
\bar{S}(e_2, e_2) &= -2\beta^2 - \beta^2 c_1^2 + 3\beta^2 c_1, \\
\bar{S}(e_3, e_3) &= -2\beta^2 + 2\beta^2 c_1, \qquad (5.3) \\
\bar{r} &= -6\beta^2 - 2\beta^2 c_1^2 + 8\beta^2 c_1.
\end{aligned}$$

Hence Theorem 3.2 is satisfied. On the other hand, from (4.5) and (5.3), we get

$$\mu + \lambda = 2\beta^2(1 - c_1).$$

Therefore, this manifold M satisfies Theorem 4.1 and Theorem 4.2. More precisely, we have

Theorem 5.1. *There exists a 3-dimensional homothetic Kenmotsu manifold with the quarter-symmetric metric connection admitting an η-Ricci soliton which is shrinking if $2\beta^2 < \mu$, steady if $2\beta^2 = \mu$, expanding if $2\beta^2 > \mu$.*

Theorem 5.2. *There exists a 3-dimensional homothetic Kenmotsu manifold with the Schouten-Van Kampen, Tanaka-Webster, or generalized Tanaka-Webster connection admitting an η-Ricci soliton which is shrinking if $\mu > 0$, steady if $\mu = 0$, expanding if $\mu < 0$.*

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Department of Mathematics, Faculty of Sciences and Arts, Erzincan Binali Yildirim University, 24100 Erzincan, Turkey
Email address: maltunbas@erzincan.edu.tr