

GENERALIZED SUBCLASSES OF k –UNIFORMLY CONVEX AND k –UNIFORMLY STARLIKE FUNCTIONS WITH VARYING ARGUMENTS

ANURADHA VISHNOI¹, ARVIND KUMAR SHARMA², SAURABH PORWAL³

ABSTRACT. The main goal of the present paper is to introduce a unified generalized subclass of k –uniformly convex and k –uniformly starlike functions with varying arguments by using fractional calculus. Coefficient bounds, extreme points, growth and distortion bounds, radii of starlikeness, close to convexity and convexity are obtained for this class. We further show that the class is closed under convolution and convex combination. Finally, we discuss a class preserving integral operator for this class. Relevant connections of the results presented here with various well-known results are briefly indicated.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

Further, we denote the subclass $\mathcal{S}$ of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) which are univalent in $\mathbb{U}$. A function $f(z)$ of the form (1.1) is said to be starlike and convex functions of order α ($0 \leq \alpha < 1$) if and only if it satisfies the analytic criteria

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \alpha$$

and

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha,$$

respectively. The classes of all starlike and convex functions of order α ($0 \leq \alpha < 1$) are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively and studied earlier by Robertson [21] and Silverman [26].

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In 1997, Bharati et al. [5] introduced the subclass k -uniformly convex functions of order α if and only if it satisfies the following condition

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq k \left| \frac{zf''(z)}{f'(z)} \right| + \alpha,$$

where $0 \leq k < \infty$, $0 \leq \alpha < 1$ and $z \in \mathbb{U}$. They [5] also introduced the class k -uniformly starlike functions of order α consisting of functions $f(z)$ of the form (1.1) if it satisfies the condition

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \alpha,$$

where $0 \leq k < \infty$, $0 \leq \alpha < 1$ and $z \in \mathbb{U}$. The classes of k -uniformly convex functions and k -uniformly starlike functions of order α are denoted by $k-\mathcal{UCV}(\alpha)$ and $k-\mathcal{ST}(\alpha)$. In fact the classes $k-\mathcal{UCV}(\alpha)$ and $k-\mathcal{ST}(\alpha)$ generalized the subclasses of convex, starlike, uniformly convex and uniformly starlike functions as follows

For $k = 0$, the subclasses $k-\mathcal{UCV}(\alpha)$ and $k-\mathcal{ST}(\alpha)$ reduce to the classes $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$, respectively studied by Robertson [21]. For $\alpha = 0$ the classes $k-\mathcal{UCV}(\alpha)$ and $k-\mathcal{ST}(\alpha)$ reduce to the classes $k-\mathcal{UCV}$ and $k-\mathcal{ST}$ studied by Kanas and Wisniowska [15, 16]. For $k = 1, \alpha = 0$ the classes $k-\mathcal{UCV}(\alpha)$ and $k-\mathcal{ST}(\alpha)$ reduce to the classes $\mathcal{UCV}$ and $\mathcal{ST}$ and studied by Goodman [12, 13] Ma and Minda [17]. It should be remarked here that these two subclasses were generalized by Dixit and Porwal [6], Gangadharan et al [11], Kanas and Srivastava [14] and Rønning [23], see also ([1, 2]). Further, the classes $k-\mathcal{UCV}(\alpha)$ and $k-\mathcal{ST}(\alpha)$ were generalized by Sim et al. [28] in the classes $k-\mathcal{UCV}(\alpha, \beta)$ and $k-\mathcal{ST}(\alpha, \beta)$ in the following way A function $f(z)$ of the form (1.1) is said to be in the class $k-\mathcal{UCV}(\alpha, \beta)$ if it satisfies the condition

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq k \left| 1 + \frac{zf''(z)}{f'(z)} - \beta \right| + \alpha,$$

Similarly, a function $f \in \mathcal{A}$ of the form (1.1) is said to be in the class $k-\mathcal{ST}(\alpha, \beta)$, if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq k \left| \frac{zf'(z)}{f(z)} - \beta \right| + \alpha,$$

Where $0 \leq \alpha < \beta \leq 1$ and $k(1 - \beta) < 1 - \alpha$. Motivated with the above mentioned work together with the work of El-Ashwah et al. [10] and Magesh [18], Aouf et al. [4] generalized these classes as follows A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{S}(\lambda, \alpha, \beta, k)$ if and only if it satisfies the condition

$$\Re \left\{ \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} \right\} \geq k \left| \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} - \beta \right| + \alpha,$$

and also in the class $k-\mathcal{UCV}(\lambda, \alpha, \beta)$, if and only if

$$\Re \left\{ \frac{zf'(z) + z^2 f''(z)}{zf'(z) + \lambda z^2 f''(z)} \right\} \geq k \left| \frac{zf'(z) + z^2 f''(z)}{zf'(z) + \lambda z^2 f''(z)} - \beta \right| + \alpha,$$

where $k \geq 0$, $-1 \leq \alpha < \beta \leq 1$, $0 \leq \lambda < 1$, $k(1 - \beta) < 1 - \alpha$ and $z \in \mathbb{U}$.

Silverman [27] (see also [8, 9] introduced the class of functions with varying argument of coefficients in the following way The class $\mathcal{V}_\eta$ of functions $f \in \mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) for which

$$\arg(a_n) = \pi + (n-1)\eta, \quad n \geq 2, \quad \eta \in R$$

and $\mathcal{V} = \bigcup_{\eta \in R} \mathcal{V}_\eta$. The class $\mathcal{V}$ is said to be the class of functions with varying arguments of coefficients. In fact for $\eta = 0$ the class $\mathcal{V}$ reduces to the class of functions with negative coefficients first introduced by Silverman [26]. The classes $\mathcal{VS}_\eta(\lambda, \alpha, \beta, k) \equiv \mathcal{S}(\lambda, \alpha, \beta, k) \cap \mathcal{V}_\eta$ and $k - \mathcal{VUCV}_\eta(\lambda, \alpha, \beta) \equiv k - \mathcal{UCV}(\lambda, \alpha, \beta) \cap \mathcal{V}_\eta$ extensively studied by Aouf et al. [4]. In the present paper motivated with the above mentioned work we introduce a unified generalized class of k -uniformly convex and k -uniformly starlike functions by using fractional calculus operator introduced by Dixit and Porwal [7]. Now, we recall the definition of fractional calculus operator which is required for this work. These definitions were found in Owa [20], Srivastava and Aouf [22].

Definition 1.1. The fractional derivative of order μ is defined for a function $f(z)$ of the form (1.1), by

$$D_z^\mu f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\mu} d\zeta, \quad (1.2)$$

where $0 \leq \mu < 1$, $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin and the multiplicity of $(z-\zeta)^{-\mu}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2. Under the hypothesis of Definition 1.1, the fractional derivative of order $n + \mu$ is defined for a function $f(z)$ by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^\mu f(z) \quad (1.3)$$

where $0 \leq \mu < 1$ and $n \in N_0 = \{0, 1, 2, \dots\}$.

For f of the form (1.1), using the Definitions 1.1 and 1.2, we introduce a new fractional derivative operator as

$$\begin{aligned} \Omega^0 f(z) &= f(z) \\ \Omega_\mu^1 f(z) &= \Gamma(1-\mu) z^{1+\mu} D_z^{1+\mu} f(z) \\ &\dots\dots\dots \\ \Omega_\mu^n f(z) &= \Omega_\mu^1 (\Omega_\mu^{n-1} f(z)). \end{aligned}$$

It is easy to see that

$$\Omega_\mu^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k, \mu)]^n a_k z^k,$$

where

$$\phi(k, \mu) = \frac{\Gamma(k+1)\Gamma(1-\mu)}{\Gamma(k-\mu)}.$$

The study of operator Ω_μ^n is of special interest because for $\mu = 0$ it reduces to the well-known and widely used Salagean operator introduced and studied by Salagean

[24]. Now, We introduce the class $\mathcal{S}(m, \mu, \lambda, \alpha, \beta, k)$ consisting of functions $f(z)$ of the form (1.1) satisfying the condition

$$\Re \left\{ \frac{\Omega_\mu^{m+1} f(z)}{(1-\lambda)\Omega_\mu^m f(z) + \lambda\Omega_\mu^{m+1} f(z)} \right\} \geq k \left| \frac{\Omega_\mu^{m+1} f(z)}{(1-\lambda)\Omega_\mu^m f(z) + \lambda\Omega_\mu^{m+1} f(z)} - \beta \right| + \alpha, \quad (1.4)$$

where $k \geq 0$, $-1 \leq \alpha < \beta \leq 1$, $0 \leq \lambda < 1$, $0 \leq \mu < 1$, $k(1-\beta) < 1-\alpha$, $m \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Further, We define

$$\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k) \equiv \mathcal{S}(m, \mu, \lambda, \alpha, \beta, k) \cap \mathcal{V}_\eta.$$

By specializing the parameters in the classes $\mathcal{S}(m, \mu, \lambda, \alpha, \beta, k)$ and $\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$ We obtain the following known subclasses studied earlier by various researchers.

- (1) $\mathcal{S}(0, 0, \lambda, \alpha, \beta, k) \equiv \mathcal{S}(\lambda, \alpha, \beta, k)$, $\mathcal{VS}_\eta(0, 0, \lambda, \alpha, \beta, k) \equiv \mathcal{VS}_\eta(\lambda, \alpha, \beta, k)$,
 $\mathcal{S}(1, 0, \lambda, \alpha, \beta, k) \equiv k\text{-}\mathcal{UCV}(\lambda, \alpha, \beta)$ and $\mathcal{VS}_\eta(1, 0, \lambda, \alpha, \beta, k) \equiv k\text{-}\nu\mathcal{UCV}_\eta(\lambda, \alpha, \beta)$
studied by Aouf et. al. [4].
- (2) $\mathcal{VS}_0(0, 0, 0, \alpha, \beta, k) \equiv k - \mathcal{ST}(\alpha, \beta)$ and $\mathcal{VS}_0(1, 0, 0, \alpha, \beta, k) \equiv k - \mathcal{UCV}(\alpha, \beta)$
studied by Sim et al. [28] and El-Ashwah et al. [10].
- (3) $\mathcal{VS}_0(0, 0, 0, \alpha, 1, k) \equiv k - \mathcal{ST}(\alpha)$ and $\mathcal{VS}_0(1, 0, 0, \alpha, 1, k) \equiv k - \mathcal{UCV}(\alpha)$ stud-
ied by Bharati et al. [5] and Shams et al. [25].
- (4) $\mathcal{VS}_0(0, 0, \lambda, \alpha, 1, k) \equiv k - \mathcal{ST}(\lambda, \alpha)$ and $\mathcal{VS}_0(1, 0, 0, 0, 1, k) \equiv k - \mathcal{UCV}(\lambda, \alpha)$
studied by Kanas and Wisniowska [15, 16] and Subramanian et al. [29].
- (5) $\mathcal{VS}_0(0, 0, 0, 0, 1, k) \equiv k - \mathcal{ST}$ and $\mathcal{VS}_0(1, 0, \lambda, \alpha, 1, k) \equiv k - \mathcal{UCV}$ studied by
Murugusundaramoorthy and Magesh [19].
- (6) $\mathcal{VS}_0(0, 0, \lambda, \alpha, 1, 0) \equiv \mathcal{T}^*(\mu, \alpha)$ and $\mathcal{VS}_0(1, 0, \lambda, \alpha, 1, 0) \equiv \mathcal{C}(\mu, \alpha)$ studied by
Altintas and Owa [3].
- (7) $\mathcal{VS}_\eta(0, 0, 0, \alpha, 1, 0) \equiv \mathcal{V}_\eta^*(\alpha)$ and $\mathcal{VS}_\eta(1, 0, 0, \alpha, 1, 0) \equiv \mathcal{VK}_\eta(\alpha)$ studied by
Silverman [27].
- (8) $\mathcal{VS}_0(0, 0, 0, \alpha, 1, 0) \equiv \mathcal{T}^*(\alpha)$ and $\mathcal{VS}_0(1, 0, 0, \alpha, 1, 0) \equiv \mathcal{C}(\alpha)$ studied by Sil-
verman [26].

In this article, we obtain a sufficient condition for functions $f(z)$ of the form (1.1) belonging to the class $\mathcal{S}(m, \mu, \lambda, \alpha, \beta, k)$. This condition is also necessary for functions belonging to the class $\mathcal{V}_\eta$. Further we obtain growth and distortion theorem, extreme points, radii of starlikeness, close to convexity, convexity, convolution condition and convex combination for the class $\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$. Finally, we discuss a class preserving integral operator for this class.

Throughout work we frequently use the notation $\epsilon_{(n, \lambda, \mu)} = 1 - \lambda + \lambda\phi(n, \mu)$.

2. MAIN RESULTS

In our first result we obtain a sufficient condition for functions $f(z)$ in the class $\mathcal{S}(m, \mu, \lambda, \alpha, \beta, k)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) and satisfy the condition*

$$\sum_{n=2}^{\infty} [\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))] |a_n| \leq 1 - \alpha - k(1 - \beta), \quad (2.1)$$

where $\phi(n, \mu) = \frac{\Gamma(n+1)\Gamma(1-\mu)}{\Gamma(n-\mu)}$, $-1 \leq \alpha < \beta \leq 1$, $0 \leq \lambda < 1$, $0 \leq \mu < 1$, $k(1-\beta) < 1-\alpha$ then $f(z)$ is in the class $\mathcal{S}(m, \mu, \lambda, \alpha, \beta, k)$.

Proof. To prove $f(z) \in \mathcal{S}(m, \mu, \lambda, \alpha, \beta, k)$ it is sufficient to prove that inequality (1.4) holds good. The condition (1.4) can be re-written as

$$\Re \left\{ (1 + ke^{i\theta}) \frac{\Omega_\mu^{m+1} f(z)}{(1-\lambda)\Omega_\mu^m f(z) + \lambda\Omega_\mu^{m+1} f(z)} - \beta ke^{i\theta} \right\} \geq \alpha. \quad (2.2)$$

For simplicity we can write

$$\Re \left\{ \frac{A(z)}{B(z)} \right\} > \alpha, \quad (2.3)$$

where

$$A(z) = (1 + ke^{i\theta})\Omega_\mu^{m+1} f(z) - \beta ke^{i\theta} ((1-\lambda)\Omega_\mu^m f(z) + \lambda\Omega_\mu^{m+1} f(z))$$

and

$$B(z) = (1-\lambda)\Omega_\mu^m f(z) + \lambda\Omega_\mu^{m+1} f(z).$$

The equivalent form of condition (2.3) can be written as

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \quad (2.4)$$

Now

$$\begin{aligned} & |A(z) + (1-\alpha)B(z)| \\ &= \left| \left((1-\beta)ke^{i\theta} + 2-\alpha \right) z - \sum_{n=2}^{\infty} \left[(\beta\epsilon_{(n,\lambda,\mu)} - \phi(n, \mu)) ke^{i\theta} - (1-\alpha)\epsilon_{(n,\lambda,\mu)} - \phi(n, \mu) \right] [\phi(n, \mu)]^m a_n z^n \right| \\ &\geq [-(1-\beta)k + 2-\alpha] |z| - \sum_{n=2}^{\infty} [(\beta\epsilon_{(n,\lambda,\mu)} - \phi(n, \mu)) k - (1-\alpha)\epsilon_{(n,\lambda,\mu)} - \phi(n, \mu)] [\phi(n, \mu)]^m |a_n| |z|^n \\ &\quad (2.5) \end{aligned}$$

and

$$\begin{aligned} & |A(z) - (1+\alpha)B(z)| \\ &= \left| \left((1-\beta)ke^{i\theta} - \alpha \right) z - \sum_{n=2}^{\infty} \left[(\phi(n, \mu) - \beta\epsilon_{(n,\lambda,\mu)}) ke^{i\theta} + \phi(n, \mu) - (1+\alpha)\epsilon_{(n,\lambda,\mu)} \right] [\phi(n, \mu)]^m a_n z^n \right| \\ &\leq [(1-\beta)k + \alpha] |z| - \sum_{n=2}^{\infty} [(\phi(n, \mu) - \beta\epsilon_{(n,\lambda,\mu)}) k - (1+\alpha)\epsilon_{(n,\lambda,\mu)} + \phi(n, \mu)] [\phi(n, \mu)]^m |a_n| |z|^n \\ &\quad (2.6) \end{aligned}$$

From (2.5) and (2.6), we have

$$\begin{aligned}
& |A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \\
& \geq 2[(1 - \alpha) - k(1 - \beta)]|z| \\
& - 2 \sum_{n=2}^{\infty} [(\phi(n, \mu) - \beta\epsilon_{(n, \lambda, \mu)})k - \alpha\epsilon_{(n, \lambda, \mu)}] [\phi(n, \mu)]^m |a_n| |z|^n \\
& > 2 \left\{ (1 - \alpha) - k(1 - \beta) - \sum_{n=2}^{\infty} [(k + 1)\phi(n, \mu) - (k\beta + \alpha)\epsilon_{(n, \lambda, \mu)}] [\phi(n, \mu)]^m |a_n| \right\}
\end{aligned}$$

The last expression is positive if condition (2.1) holds.

This completes the proof of Theorem 2.1. $\square$

In our next result we shall show that the condition (2.1) is also necessary for functions $f(z)$ of the form (1.1) and in the class $\mathcal{V}_\eta$.

Theorem 2.2. *Let $f(z)$ be of the form (1.1) and in $\mathcal{V}_\eta$, then $f \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$ if and only if condition (2.1) is satisfied.*

Proof. The proof of "if part" follows from Theorem 2.1. Hence, we need only to prove the "only if" part of the theorem. If $f(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$ then by condition (1.4) we have

$$\begin{aligned}
& \Re \left\{ \frac{1 - \alpha + \sum_{n=2}^{\infty} [\phi(n, \mu) - \alpha\epsilon_{(n, \lambda, \mu)}] [\phi(n, \mu)]^m a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \epsilon_{(n, \lambda, \mu)} [\phi(n, \mu)]^m a_n z^{n-1}} \right\} \geq \\
& k \left| \frac{1 - \beta + \sum_{n=2}^{\infty} [\phi(n, \mu) - \beta\epsilon_{(n, \lambda, \mu)}] [\phi(n, \mu)]^m a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \epsilon_{(n, \lambda, \mu)} [\phi(n, \mu)]^m a_n z^{n-1}} \right|.
\end{aligned}$$

Since f is a function of the form (1.1) with the argument property given in the class $\mathcal{V}_\eta$ and setting $z = re^{i\eta}$ in the above inequality, we obtain

$$\begin{aligned}
& \frac{1 - \alpha - \sum_{n=2}^{\infty} [\phi(n, \mu) - \alpha\epsilon_{(n, \lambda, \mu)}] [\phi(n, \mu)]^m |a_n| r^{n-1}}{1 + \sum_{n=2}^{\infty} \epsilon_{(n, \lambda, \mu)} [\phi(n, \mu)]^m |a_n| r^{n-1}} \geq \\
& k \frac{1 - \beta + \sum_{n=2}^{\infty} [\phi(n, \mu) - \beta\epsilon_{(n, \lambda, \mu)}] [\phi(n, \mu)]^m |a_n| r^{n-1}}{1 + \sum_{n=2}^{\infty} \epsilon_{(n, \lambda, \mu)} [\phi(n, \mu)]^m |a_n| r^{n-1}}. \quad (2.7)
\end{aligned}$$

Making $r \rightarrow r^{-1}$ in (2.7), we have

$$\sum_{n=2}^{\infty} [\phi(n, \mu)]^m [(k + 1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))] |a_n| \leq 1 - \alpha - k(1 - \beta).$$

The result is sharp for the function

$$f_{n, \eta}(z) = z - \frac{[1 - \alpha - k(1 - \beta)] e^{i(1-n)\eta}}{[\phi(n, \mu)]^m [(k + 1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]} z^n, \quad (2.8)$$

where $n \geq 2$, $0 \leq \eta < 2\pi$. $\square$

Corollary 2.3. Let $f(z)$ be of the form (1.1) and in the class $\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$. Then

$$|a_n| \leq \frac{1 - \alpha - k(1 - \beta)}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}, \quad n \geq 2. \quad (2.9)$$

The result is sharp for the function defined by (2.8).

Theorem 2.4. Let $f(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$, then for $|z| < r = 1$ we have

$$|f(z)| \leq r + \frac{1 - \alpha - k(1 - \beta)}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]} r^2 \quad (2.10)$$

and

$$|f(z)| \geq r - \frac{1 - \alpha - k(1 - \beta)}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]} r^2 \quad (2.11)$$

Proof. We only prove the right inequality. The proof for left hand inequality is similar so we will omit the details. Let $f(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$, taking absolute value we have

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \\ &= r + r^2 \frac{1 - \alpha - k(1 - \beta)}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]} \\ &\quad \sum_{n=2}^{\infty} \frac{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]}{1 - \alpha - k(1 - \beta)} |a_n| \\ &= r + r^2 \frac{1 - \alpha - k(1 - \beta)}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]} \\ &\quad \sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} |a_n| \\ &= r + \frac{1 - \alpha - k(1 - \beta)}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]} r^2. \end{aligned}$$

The results are sharp for the function

$$f(z) = z - \frac{[1 - \alpha - k(1 - \beta)] e^{-i\eta}}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]} z^2. \quad (2.12)$$

□

Theorem 2.5. Let $f(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$, then for $|z| < r = 1$ we have

$$|f'(z)| \leq 1 + \frac{2(1 - \alpha - k(1 - \beta))}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha)\left(1 - \lambda + \frac{2}{1-\mu}\lambda\right)\right]} r \quad (2.13)$$

and

$$|f'(z)| \geq 1 - \frac{2(1 - \alpha - k(1 - \beta))}{\left(\frac{2}{1-\mu}\right)^m \left[\frac{2}{1-\mu}(k+1) - (k\beta + \alpha) \left(1 - \lambda + \frac{2}{1-\mu}\lambda\right) \right]} r. \quad (2.14)$$

The results are sharp for the function (2.12).

Proof. The proof of above theorem is much akin to previous theorem so we omit the detail. $\square$

Theorem 2.6. Let $f(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$ with argument property as in the class $\mathcal{V}_\eta$. Define $f_{1,\eta}(z) = z$, and

$$f_{n,\eta}(z) = z - \frac{[1 - \alpha - k(1 - \beta)] e^{i(1-n)\eta}}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]} z^n, \quad (2.15)$$

$$0 \leq \eta < 2\pi, \quad n \geq 2.$$

Then $f(z)$ is in the class $\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} x_n f_{n,\eta}(z), \quad (2.16)$$

where $x_n \geq 0$ ($n \geq 1$) and $\sum_{n=1}^{\infty} x_n = 1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} x_n f_{n,\eta}(z) = x_1 f_{1,\eta}(z) + \sum_{n=2}^{\infty} x_n f_{n,\eta}(z) \\ &= \left(1 - \sum_{n=2}^{\infty} x_n\right) z + \sum_{n=2}^{\infty} x_n \left[z - \frac{[1 - \alpha - k(1 - \beta)] e^{i(1-n)\eta}}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]} z^n \right] \\ &= z - \sum_{n=2}^{\infty} \frac{[1 - \alpha - k(1 - \beta)] e^{i(1-n)\eta}}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]} x_n z^n. \end{aligned}$$

From Theorem 2.2 we have

$$\begin{aligned} &\sum_{n=2}^{\infty} [\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))] |a_n| \\ &= \sum_{n=2}^{\infty} [\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))] \\ &\quad \frac{[1 - \alpha - k(1 - \beta)] e^{i(1-n)\eta}}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]} \\ &\leq 1 - \alpha - k(1 - \beta). \end{aligned}$$

Conversely, suppose that $f(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$. Set

$$x_n = \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} |a_n|, \quad n \geq 2.$$

Then by Theorem 2.2 we have $0 \leq x_n \leq 1, (n \geq 2)$, we define

$$x_1 = 1 - \sum_{n=2}^{\infty} x_n,$$

then by Theorem 2.2 we have $x_1 \geq 0$. Consequently, we obtain

$$f(z) = \sum_{n=1}^{\infty} x_n f_{n,\eta}(z).$$

This completes the proof of theorem. $\square$

In our next result we obtain the radius of close-to-convexity for the class $\mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$.

Theorem 2.7. *Let $f(z) \in \mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$. Then $f(z)$ is close-to-convex of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf \left[\frac{(1 - \sigma) \{ [\phi(n, \mu)]^m [(k + 1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))] \}}{1 - \alpha - k(1 - \beta)} \right]^{\frac{1}{n-1}}, \quad (2.17)$$

$n \geq 2$.

The result is sharp for the function $f(z)$ is defined in (2.8).

Proof. Let $f(z) \in \mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$ and $f(z)$ is close-to-convex of order σ , we have

$$|f'(z) - 1| < 1 - \sigma. \quad (2.18)$$

On substituting the value of $f(z)$ given by (1.1) in L.H.S. of (2.18) we obtain

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n |a_n| |z|^{n-1}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n}{1 - \sigma} |a_n| |z|^{n-1} < 1.$$

Since $f(z) \in \mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$, from Theorem 2.2 we have

$$\sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k + 1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} |a_n| \leq 1.$$

The condition (2.18) holds good if

$$\frac{n}{1 - \sigma} |z|^{n-1} \leq \frac{[\phi(n, \mu)]^m [(k + 1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)}$$

or, equivalently

$$|z|^{n-1} = \left[\frac{(1 - \sigma) \{ [\phi(n, \mu)]^m [(k + 1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))] \}}{1 - \alpha - k(1 - \beta)} \right],$$

which gives the desired result and thus the proof of this theorem is established. $\square$

Applying the same technique of Theorem 2.7, we only state the results of following theorem.

Theorem 2.8. *Let $f(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$. Then*

(1) *$f(z)$ is starlike order σ ($0 \leq \sigma < 1$) in the disc $|z| < r_2$, where*

$$r_2 = \inf \left[\left(\frac{1-\sigma}{n-\sigma} \right) \frac{\{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]\}}{1 - \alpha - k(1 - \beta)} \right]^{\frac{1}{n-1}}, \quad (2.19)$$

$$n \geq 2.$$

(2) *$f(z)$ is convex order σ ($0 \leq \sigma < 1$) in the disc $|z| < r_3$, where*

$$r_3 = \inf \left[\left(\frac{1-\sigma}{n(n-\sigma)} \right) \frac{\{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]\}}{1 - \alpha - k(1 - \beta)} \right]^{\frac{1}{n-1}}, \quad (2.20)$$

$$n \geq 2.$$

The results given in (2.19) and (2.20) are sharp for the function $f(z)$ given in (2.8).

In our next theorem we determine convolution properties for the class $\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$. For this purpose we recall the definition of modified Hadamard product which is given below

Let the functions $f_j(z)$ ($j = 1, 2$) be defined by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,i} \geq 0; j = 1, 2. \quad (2.21)$$

Then we define the modified Hadamard product of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \quad (2.22)$$

Now, we prove the following result

Theorem 2.9. *Let $f_j(z)$ ($j = 1, 2$) be defined by (2.21) is in the class $\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$, where $-1 \leq \alpha < \beta \leq 1$, $0 \leq \lambda < 1$, $0 \leq \mu < 1$, $k(1 - \beta) < 1 - \alpha$, $m \in \mathbb{N}_0$ and $z \in \mathbb{U}$. Then $f_1 * f_2(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \delta_1, \beta, k)$ for*

$$\delta_1 = \frac{X}{Y}, \quad (2.23)$$

where

$$\begin{aligned} X &= \left[\{\phi(2, \mu)\}^m \left[\phi(2, \mu)(k+1) - (k\beta + \alpha) \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] \right]^2 - \\ &\quad \{\phi(2, \mu)\}^m \left[\phi(2, \mu)(k+1) - k\beta \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] (1 - \alpha - k(1 - \beta))^2, \end{aligned} \quad (2.24)$$

$$Y = \left[\{\phi(2, \mu)\}^m \left[\phi(2, \mu)(k+1) - (k\beta + \alpha) \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] \right]^2 - \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) (1 - \alpha - k(1 - \beta))^2 \quad (2.25)$$

and $\phi(2, \mu) = \frac{2}{1-\mu}$. The result is sharp.

Proof. Here we have to find largest δ_1 such that

$$\sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \delta_1)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \delta_1 - k(1 - \beta)} |a_{n,1}a_{n,2}| \leq 1. \quad (2.26)$$

Since $f_j(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$ ($j = 1, 2$) from Theorem 2.2 and applying Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} \sqrt{|a_{n,1}a_{n,2}|} \leq 1. \quad (2.27)$$

Thus it is sufficient to prove that

$$\begin{aligned} & \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \delta_1)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \delta_1 - k(1 - \beta)} |a_{n,1}a_{n,2}| \leq \\ & \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} \sqrt{|a_{n,1}a_{n,2}|}, n \geq 2. \end{aligned} \quad (2.28)$$

Or, equivalently

$$\sqrt{|a_{n,1}a_{n,2}|} \leq \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \delta_1)(1 - \lambda + \lambda\phi(n, \mu))]} \frac{(1 - \delta_1 - k(1 - \beta))}{(1 - \alpha - k(1 - \beta))}, n \geq 2. \quad (2.29)$$

From (2.27) we have

$$\sqrt{|a_{n,1}a_{n,2}|} \leq \frac{1 - \alpha - k(1 - \beta)}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}, n \geq 2. \quad (2.30)$$

Consequently, we only need to prove that

$$\begin{aligned} & \frac{1 - \alpha - k(1 - \beta)}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]} \leq \\ & \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \delta_1)(1 - \lambda + \lambda\phi(n, \mu))]} \frac{(1 - \delta_1 - k(1 - \beta))}{(1 - \alpha - k(1 - \beta))} \end{aligned} \quad (2.31)$$

The above expression is equivalent to

$$\begin{aligned} \delta_1 = & \frac{\left[\{\phi(n, \mu)\}^m \left[\phi(n, \mu)(k+1) - (k\beta + \alpha) \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] \right]^2 - \{\phi(n, \mu)\}^m \left[\phi(n, \mu)(k+1) - k\beta \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] (1 - \alpha - k(1 - \beta))^2}{\left[\{\phi(n, \mu)\}^m \left[\phi(n, \mu)(k+1) - (k\beta + \alpha) \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] \right]^2 - \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) (1 - \alpha - k(1 - \beta))^2} \\ & = \Delta(n). \end{aligned} \quad (2.32)$$

Since $\Delta(n)$ is an increasing function of n ($n \geq 2$), letting $n = 2$ in (2.32) we obtain

$$\begin{aligned} \delta_1 = \Delta(2) = & \frac{\left[\{\phi(2, \mu)\}^m \left[\phi(2, \mu)(k+1) - (k\beta + \alpha) \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] \right]^2 - \{\phi(2, \mu)\}^m \left[\phi(2, \mu)(k+1) - k\beta \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] (1 - \alpha - k(1 - \beta))^2}{\left[\{\phi(2, \mu)\}^m \left[\phi(2, \mu)(k+1) - (k\beta + \alpha) \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) \right] \right]^2 - \left(1 - \lambda + \frac{2}{1-\mu} \lambda \right) (1 - \alpha - k(1 - \beta))^2}, \end{aligned} \quad (2.33)$$

or, equivalently

$$\delta_1 = \Delta(2) = \frac{X}{Y}, \quad (2.34)$$

where X and Y are given by (2.24) and (2.25), respectively. which proves the desired result.

The result is sharp for the function $f(z)$ given by (2.8). $\square$

In our next result we show that the class $\mathcal{VS}_\eta(m, \mu, \lambda, \delta_1, \beta, k)$ is closed under convolution.

Theorem 2.10. *The class $\mathcal{VS}_\eta(m, \mu, \lambda, \alpha, \beta, k)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \dots$, $f_i(z) \in \mathcal{VS}_\eta(m, \mu, \lambda, \delta_1, \beta, k)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n. \quad (2.35)$$

Then by Theorem 2.2 we have

$$\sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} |a_{n,i}| \leq 1. \quad (2.36)$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{n,i} \right) z^n. \quad (2.37)$$

Then by Theorem 2.2 we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} \left(\sum_{i=1}^{\infty} t_i a_{n,i} \right) \\ &= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} a_{n,i} \right) \\ &\leq \sum_{i=1}^{\infty} t_i \\ &= 1. \end{aligned}$$

Therefore $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$.

Thus, we say that the class $\mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$ is closed under convex combination. $\square$

Remark 2.1. (1) If we put $\mu = 0, m = 0, m = 1$ in Theorems 2.1-2.10 then we obtain the corresponding results of Aouf et al. [4].

(2) If we put $\mu = 0, \lambda = 0, m = 0, m = 1, \eta = 0$ in Theorems 2.1-2.10 then we obtain the corresponding results of Sim et al. [28].

(3) If we put $\mu = 0, \lambda = 0, m = 0, m = 1, \beta = 1, \eta = 0$ in Theorems 2.1-2.10 then we obtain the corresponding results of Bharti et al. [5].

(4) If we put $\mu = 0, \lambda = 0, m = 0, m = 1, \alpha = 0, \beta = 1, \eta = 0$ in Theorems 2.1-2.10 then we obtain the corresponding results of Kanas and Wisniowska [15, 16].

3. A FAMILY OF CLASS PRESERVING INTEGRAL OPERATOR

Let $f(z) \in \mathcal{A}$ be of the form (1.1) then the integral operator $F(z)$ given by the relation

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (c > -1). \quad (3.1)$$

Theorem 3.1. Let $f(z) \in \mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$. Then $F(z)$ defined by the relation (3.1) is also belongs to the class $\mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$.

Proof. Since $f(z) \in \mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$ then by Theorem 2.2 we have

$$\sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} |a_n| \leq 1. \quad (3.2)$$

By definition of $F(z)$ given in (3.1) we can express $F(z)$ as

$$F(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n. \quad (3.3)$$

Now

$$\begin{aligned}
& \sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} \left| \frac{c+1}{c+n} a_n \right| \\
& \leq \sum_{n=2}^{\infty} \frac{[\phi(n, \mu)]^m [(k+1)\phi(n, \mu) - (k\beta + \alpha)(1 - \lambda + \lambda\phi(n, \mu))]}{1 - \alpha - k(1 - \beta)} |a_n| \\
& \leq 1.
\end{aligned}$$

Thus $F(z) \in \mathcal{VS}_{\eta}(m, \mu, \lambda, \alpha, \beta, k)$.

Thus the proof of Theorem 3.1 is established. $\square$

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^{1,2} DEPARTMENT OF MATHEMATICS, JANTA MAHAVIDYALAYA, AJITMAL, AURAIYA-206121, UTTAR PRADESH, INDIA

¹ [HTTPS://ORCID.ORG/0009-0004-2283-0934](https://orcid.org/0009-0004-2283-0934)

² [HTTPS://ORCID.ORG/0009-0000-5351-5241](https://orcid.org/0009-0000-5351-5241)

³ DEPARTMENT OF MATHEMATICS, RAM SAHAI GOVERNMENT DEGREE COLLEGE, BAIRI, SHIVRAJPUR-209205, KANPUR, UTTAR PRADESH, INDIA

[HTTPS://ORCID.ORG/0000-0003-0847-3550](https://orcid.org/0000-0003-0847-3550)

Email address: ¹ anuvishnoi16@gmail.com, ² aksharma.maths@gmail.com,

³ saurabhjcb@rediffmail.com