

ON THE TOPOLOGICAL DYNAMICS OF MAPPINGS OF RANDOM VARIABLES

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ABSTRACT. The aim of this paper is to investigate certain topological discrete dynamical properties of mappings defined on a random normed space of random variables, where the norm distribution is modelled via a nonnegative random variable. In this context, we introduce stochastic counterparts of some of the well known concepts such as stability, sink, source, expansiveness, etc. studied in classical discrete dynamical systems, and establish some basic facts. Such an approach could provide a natural probabilistic setting in scientific applications which produce random data or measurement errors due to randomness. Our basic tools are the iteration of mappings of random variables and convergence in probability.

1. INTRODUCTION

This work is based on a great area of research, namely, the theory of topological dynamics which interacts with a lot of mathematical disciplines such as algebra, measure theory, differential equations, etc. Topological dynamical systems investigate the long term behaviour of continuous mappings defined on topological spaces, in general. There are many papers and books in this area, and for a good review and the history of the subject, we refer to [2], [10] and [32]. For more recent developments, see e.g. [8].

It is well known that in real world applications, the evolution of a dynamical system may not be deterministic and thus it can most likely produce random behaviour. In this context, of particular interest in the probabilistic part of the theory of dynamical systems is the concept of iteration of functions of random variables and iteration of random functions, which have many applications in probability theory. For instance, certain developments in this direction can be found e.g. in [15], [17] and [25], where hierarchical models are considered. In [33], the strong law of large numbers for a class of such iterations is considered. Similarly, for the iteration of random functions which has applications in Markov processes, see e.g. [1]; and for applications in queuing theory, see [9].

In the current paper, we will also investigate the concept of iteration of functions of random variables but via a different method, that is, by using the theory of

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random normed (RN) spaces. In this context, we will introduce some probabilistic counterparts of some of the well known concepts of classical dynamics; such as stability, sink, attractivity, etc., and try to establish certain basic facts.

Now let us outline what we have done in our work. First, our phase space will be an RN space (in fact, an RN space of real random variables). This is a vector space which is a special type of a random metric (RM) space that was first introduced by Špaček ([27, 28]) and further investigated by many others (see, e.g. [4], [11], and [23]). In an RN space, the norm of a vector is random and thus it is modelled by a nonnegative random variable (see, e.g. [11]). In particular, an RN space of random variables is also a topological vector space (see e.g. [31]). Now closely related with the theory of RN spaces is the theory of probabilistic metric (PM) spaces. In a PM space, the distance between two points is modelled by a probability distribution function (see e.g. [23]). Certain attempts in the direction of fixed point theory and iteration of mappings defined on PM spaces can be found in the monographs [7] and [13]. In our paper, we will investigate certain topological dynamical properties of mappings defined on an RN space of random variables, and our time variable will be discrete. Hence the evolution of a discrete dynamical system which contains random data may be handled via the tools of RN space of random variables. Namely, the random inputs driven into the dynamical system produce random outputs via the iteration of a mapping f of random variables. Thus the long term behaviour of the system can be modelled by the tools of RN spaces and convergence in probability.

In what follows, we first recall some basic notions related to RN spaces. In the third section, based on certain concepts and results in classical topological dynamics (e.g. given in [3], [19], [26] and [32]), we will introduce the probabilistic (random) counterparts of these concepts and prove some basic facts.

2. PRELIMINARIES

First, we recall some of the basic concepts related to the theory of RM and RN spaces, and we refer to e.g. [11] and [12] for more details.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathbb{R}$ denote the set of all real numbers. Here are some notations [we identify almost surely (*a.s.*) equal functions]:

(i) $L_0(\Omega, \mathbb{R})$: The set of equivalence classes of real valued random variables defined on Ω ,

(ii) $L_0^+(\Omega)$: The set of all $\xi \in L_0(\Omega, \mathbb{R})$ satisfying $\xi(\omega) \geq 0$ *a.s.*

The zero element in the linear space $L_0(\Omega, \mathbb{R})$ will be denoted by Θ , that is, $\Theta(\omega) = 0$ *a.s.* Note that, for any $\xi, \eta \in L_0(\Omega, \mathbb{R})$, the relation $\xi \preceq \eta$ means that, $\xi(\omega) \leq \eta(\omega)$ *a.s.* Thus $L_0(\Omega, \mathbb{R})$ is a Dedekind complete vector lattice (see [22]).

Definition 2.1. [11] An ordered pair $(S, \mathcal{X})$ is called a *random metric space* (briefly, an RM space) with base $(\Omega, \mathcal{A}, P)$ if S is a nonempty set and $\mathcal{X}$ is a mapping from $S \times S$ to $L_0^+(\Omega)$ such that, denoting $\mathcal{X}(p, q)$ by X_{pq} , for all p, q, r in S , the following hold:

- (RM-1) $X_{pq} = \Theta$ iff $p = q$,
- (RM-2) $X_{pq} = X_{qp}$,
- (RM-3) $X_{pr} \preceq X_{pq} + X_{qr}$.

X_{pq} is said to be the *random distance* between p and q . Moreover, given $D \subset S$, the *random distance* between D and a point $p \in S$ is defined by $X_{pD} = \inf \{X_{pq} : q \in D\}$.

Now let $(S, \mathcal{X})$ be an RM space with base $(\Omega, \mathcal{A}, P)$. Given a point p in S , $\varepsilon > 0$ and $\lambda \in (0, 1)$, the set

$$\mathcal{N}_p(\varepsilon, \lambda) = \{q \in S : P(X_{pq} < \varepsilon) > 1 - \lambda\}$$

is called the (ε, λ) -*neighbourhood* of p (see also [23]). Thus our RM space will be endowed with the (ε, λ) -*topology* which is generated by the (ε, λ) -neighbourhoods. Hence an RM space is Hausdorff, uniform and metrizable. Moreover, if (p_n) is a sequence in $(S, \mathcal{X})$ and $p \in S$, then (p_n) is convergent to p in the (ε, λ) -topology iff $(X_{p_n p})$ is convergent in probability to Θ . In this case, we will write $P\text{-}\lim X_{p_n p} = \Theta$.

Furthermore, the mapping $\mathcal{X}$ is uniformly continuous. Thus given $\varepsilon > 0$ and $\lambda \in (0, 1)$ we have $P(X_{pr} < \varepsilon) > 1 - \lambda$ whenever $P(X_{pq} < \frac{\varepsilon}{2}) > 1 - \frac{\lambda}{2}$ and $P(X_{qr} < \frac{\varepsilon}{2}) > 1 - \frac{\lambda}{2}$ where $p, q, r \in S$ (see [4]).

Now we recall the definition of an RN space (see also [14]):

Definition 2.2. [11] Let S be a linear space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), with a zero vector θ and $\mathbf{N}: S \rightarrow L_0^+(\Omega)$ be a mapping. Then, the ordered pair $(S, \mathbf{N})$ is called a *random normed space* (briefly, an RN space) over $\mathbb{K}$ with base $(\Omega, \mathcal{A}, P)$ if the following are satisfied (denoting $\mathbf{N}(p)$ by $\mathbf{N}_p$ for any p in S):

(RN-1) $\mathbf{N}_p = \Theta$ iff $p = \theta$,

(RN-2) $\mathbf{N}_{\alpha p} = |\alpha| \mathbf{N}_p$ for each $\alpha \in \mathbb{K}$ and $p \in S$,

(RN-3) $\mathbf{N}_{p+q} \preceq \mathbf{N}_p + \mathbf{N}_q$ for any $p, q \in S$.

Note that every RN space $(S, \mathbf{N})$ induces an RM space $(S, \mathcal{X})$ if we define $\mathcal{X}(p, q) = X_{pq} = \mathbf{N}_{p-q}$ for every $p, q \in S$.

Example 2.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(V, \|\cdot\|)$ a separable normed space. Let S be the set of equivalence classes of measurable mappings from Ω into V , where V is endowed with a Borel σ -algebra. Consider the mapping $\mathbf{N}$ from S into $L_0^+(\Omega)$ defined by

$$[\mathbf{N}(p)](\omega) = \mathbf{N}_p(\omega) = \|p(\omega)\|$$

for every $p \in S$ and $\omega \in \Omega$. Then $(S, \mathbf{N})$ is an RN space with base $(\Omega, \mathcal{A}, P)$ (see [12], and also [20], [24] and [29]).

Note that, by Example 2.1, given a probability space $(\Omega, \mathcal{A}, P)$, the set $L_0(\Omega, \mathbb{R})$ is a special RN space with the random norm $\mathbf{N}$ defined by $\mathbf{N}_p(\omega) = |p(\omega)|$ where $p \in L_0(\Omega, \mathbb{R})$. Throughout the rest of the paper, we will base ourselves on this special RN space and denote its elements (random variables) by ξ, η , etc. Note that $L_0(\Omega, \mathbb{R})$ will always be endowed with the (ε, λ) -topology induced by the random norm. In this case, it can be easily seen that a sequence (ξ_n) in $L_0(\Omega, \mathbb{R})$ is convergent to a random variable ξ with respect to the (ε, λ) -topology iff $P\text{-}\lim \mathbf{N}_{\xi_n - \xi} = \Theta$, or equivalently, $P\text{-}\lim \xi_n = \xi$. Hence the (ε, λ) -topology is a natural topological setting for convergence in probability in $L_0(\Omega, \mathbb{R})$.

3. MAIN RESULTS

In what follows, to overcome the measurability problems, $(\Omega, \mathcal{A}, P)$ will be assumed to be a complete probability space. Moreover, our phase space will be

$L_0(\Omega, \mathbb{R})$ or a subset B of $L_0(\Omega, \mathbb{R})$, and any $f : B \rightarrow B$ will denote a continuous mapping. Given an n th iteration f^n of f , we will denote the identity mapping by f^0 . The pair $(L_0(\Omega, \mathbb{R}), f)$ or (B, f) will denote the topological dynamical system governed by f . A fixed point of f will be denoted by ξ^* .

First, we will prove a simple result related to fixed points.

Theorem 3.1. *Let $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ be given and suppose that $P - \lim f^n(\xi_0) = \xi^*$ for some $\xi_0 \in L_0(\Omega, \mathbb{R})$. Then $f(\xi^*) = \xi^*$.*

Proof. Since $f^{n+1}(\xi_0) = f(f^n(\xi_0))$ and f is continuous, we have

$$P - \lim f(f^n(\xi_0)) = f(\xi^*)$$

while $P - \lim f^{n+1}(\xi_0) = \xi^*$, which completes the proof. $\square$

Now we will obtain a finite family of equivalent random metrics on the space $L_0(\Omega, \mathbb{R})$ via the iterations of f .

Theorem 3.2. *Let $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ be given and $\mathcal{X}$ be the random metric defined by $\mathcal{X}(\xi, \eta) = X_{\xi\eta} = \mathbf{N}_{\xi-\eta}$ for every $\xi, \eta \in L_0(\Omega, \mathbb{R})$. For each $\xi, \eta \in L_0(\Omega, \mathbb{R})$ and $n \in \mathbb{N}$, define*

$$\mathbf{N}_{\xi-\eta}^{(n)} = X_{\xi\eta}^{(n)} = \sup_{0 \leq k \leq n-1} \mathbf{N}_{f^k(\xi)-f^k(\eta)},$$

where $\mathbf{N}^{(1)} = \mathbf{N}$ and $X^{(1)} = X$. Then each $X^{(n)}$ is a random metric on $L_0(\Omega, \mathbb{R})$ and all $X^{(n)}$ are equivalent.

Proof. First, let us show that each $X^{(n)}$ is a random metric:

(RM-1) $X_{\xi\eta}^{(n)} = \Theta$ iff $\mathbf{N}_{f^k(\xi)-f^k(\eta)} = \Theta$ for each $k = 0, \dots, n-1$ iff $f^k(\xi) - f^k(\eta) = \Theta$ for each k , iff $\xi = \eta$.

(RM-2) Straightforward.

(RM-3) Let $\xi, \eta, \zeta \in L_0(\Omega, \mathbb{R})$. Then, for each $k = 0, \dots, n-1$ we have

$$\mathbf{N}_{f^k(\xi)-f^k(\zeta)} \preceq \mathbf{N}_{f^k(\xi)-f^k(\eta)} + \mathbf{N}_{f^k(\eta)-f^k(\zeta)}$$

which implies that

$$\mathbf{N}_{f^k(\xi)-f^k(\zeta)} \preceq \sup_{0 \leq k \leq n-1} \mathbf{N}_{f^k(\xi)-f^k(\eta)} + \sup_{0 \leq k \leq n-1} \mathbf{N}_{f^k(\eta)-f^k(\zeta)}$$

for each k , and thus we get $\sup_{0 \leq k \leq n-1} \mathbf{N}_{f^k(\xi)-f^k(\zeta)} = X_{\xi\zeta}^{(n)} \preceq X_{\xi\eta}^{(n)} + X_{\eta\zeta}^{(n)}$.

Now let us prove that the random metric $X^{(n)}$ is equivalent to X for each $n \in \mathbb{N}$. Let $\xi, \eta \in L_0(\Omega, \mathbb{R})$. Then we have $X_{\xi\eta} \preceq X_{\xi\eta}^{(n)}$. On the other hand, if $\varepsilon > 0$, $\lambda \in (0, 1)$ and $n \in \mathbb{N}$, then by the continuity of the mappings $f^0, f, \dots, f^{n-1}$ at ξ , there exist a $\delta_k = \delta_k(\xi, \varepsilon) > 0$ and an $\alpha_k = \alpha_k(\xi, \lambda) \in (0, 1)$ such that for every $\eta \in L_0(\Omega, \mathbb{R})$ with $P(X_{\xi\eta} < \delta_k) > 1 - \alpha_k$, we get $P(X_{f^k(\xi)f^k(\eta)} < \varepsilon) > 1 - \lambda$ for each $k = 0, 1, \dots, n-1$. Now let $\delta = \min\{\delta_k : k = 0, 1, \dots, n-1\}$ and $\alpha = \max\{\alpha_k : k = 0, 1, \dots, n-1\}$. In this case, $X_{\xi\eta} < \delta$ implies $X_{\xi\eta} < \delta_k$ for each k , which gives

$$P(X_{\xi\eta} < \delta_k) \geq P(X_{\xi\eta} < \delta) > 1 - \alpha \geq 1 - \alpha_k$$

for each k , if $P(X_{\xi\eta} < \delta) > 1 - \alpha$. Thus we have $P(X_{f^{(k)}(\xi)f^{(k)}(\eta)} < \varepsilon) > 1 - \lambda$ for each k whenever $P(X_{\xi\eta} < \delta) > 1 - \alpha$, and thus $P(X_{\xi\eta}^{(n)} < \varepsilon) > 1 - \lambda$ when $P(X_{\xi\eta} < \delta) > 1 - \alpha$. Hence we can write $\mathcal{N}_{\xi}^{(1)}(\delta, \alpha) = \mathcal{N}_{\xi}(\delta, \alpha) \subset \mathcal{N}_{\xi}^{(n)}(\varepsilon, \lambda) \subset \mathcal{N}_{\xi}^{(1)}(\varepsilon, \lambda) = \mathcal{N}_{\xi}(\varepsilon, \lambda)$ for every $\xi \in L_0(\Omega, \mathbb{R})$, where $\mathcal{N}_{\xi}^{(n)}(\varepsilon, \lambda)$ denotes the (ε, λ) -neighbourhood of ξ with respect to the random metric $X^{(n)}$. $\square$

Now we will study some of the basic concepts of classical topological dynamics in the setting of $(L_0(\Omega, \mathbb{R}), f)$. We begin with

Definition 3.1. Let ξ^* be a fixed point of f . Then we say that ξ^* is *stable in probability* provided that, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exist a $\delta > 0$ and $\alpha \in (0, 1)$ such that $P(\mathbf{N}_{\xi-\xi^*} < \delta) > 1 - \alpha$ implies that $P(\mathbf{N}_{f^n(\xi)-\xi^*} < \varepsilon) > 1 - \lambda$ for all $n \in \mathbb{N}$.

Example 3.3. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ be defined by $f(\xi) = 2 - \xi$. Then we have the only fixed point ξ^* defined by $\xi^*(\omega) = 1$ a.s. Now let any $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given. Then for any $\xi \in L_0(\Omega, \mathbb{R})$, we have $\mathbf{N}_{f^n(\xi)-\xi^*}(\omega) = \mathbf{N}_{\xi-\xi^*}(\omega)$ a.s. for all $n \in \mathbb{N}$. Hence it is enough to choose $\delta = \varepsilon$ and $\alpha = \lambda$. Thus ξ^* is stable in probability.

Remark 3.1. For the concept of stability in probability for continuous dynamical systems, the reader may refer to, e.g. [16].

Now according to Definition 3.1, given a fixed point ξ^* of f , we say that ξ^* is *unstable in probability* provided that, there exist an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that for every $\delta > 0$ and $\alpha \in (0, 1)$ we can find a $\xi \in L_0(\Omega, \mathbb{R})$ with the following properties:

- (i) $P(\mathbf{N}_{\xi-\xi^*} < \delta) > 1 - \alpha$
- (ii) There exists an $N \in \mathbb{N}$ such that $P(\mathbf{N}_{f^N(\xi)-\xi^*} < \varepsilon) \leq 1 - \lambda$.

Example 3.4. Let $(\Omega, \mathcal{A}, P)$ be the Lebesgue probability space where $\Omega = [0, 1]$ and $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ be defined by $f(\xi) = 2\xi$. Then the only fixed point is $\xi^* = \Theta$. Now let $\varepsilon = 1$ and $\lambda = \frac{1}{2}$. Given any $\delta > 0$ and $\alpha \in (0, 1)$, choose a $\xi \in L_0(\Omega, \mathbb{R})$ such that $0 < \xi(\omega) \leq \delta$ a.s. For instance, if $\delta \geq 1$, then define $\xi(\omega) = \omega$. Otherwise, let $\xi(\omega) = \delta_0$ a.s., where δ_0 is fixed and $0 < \delta_0 < \delta$. Then, for each case of δ and corresponding ξ , we have $P(\mathbf{N}_{\xi-\xi^*} < \delta) > 1 - \alpha$ and we can find a large enough $N \in \mathbb{N}$ such that $P(\xi < \frac{1}{2^N}) \leq 1 - \lambda$. Thus ξ^* is unstable in probability.

Now we will consider the limit points and stability conditions for an orbit of an initial condition $\xi_0 \in L_0(\Omega, \mathbb{R})$. As in any topological dynamical system, let $\mathcal{O}(\xi_0)$ be the orbit of ξ_0 , i.e., $\mathcal{O}(\xi_0) = \{\xi_0, f(\xi_0), f^2(\xi_0), \dots\}$. Then a $\xi \in L_0(\Omega, \mathbb{R})$ will be called a *limit point* of $\mathcal{O}(\xi_0)$ if there exists a subsequence $\{\xi_{n_k} : k \in \mathbb{N}\}$ of $\mathcal{O}(\xi_0)$ such that $P - \lim \mathbf{N}_{\xi_{n_k}-\xi} = \Theta$. The set of all limit points of the orbit $\mathcal{O}(\xi_0)$ will be denoted by $\mathcal{L}(\xi_0)$.

Definition 3.2. Let $\xi_0 \in L_0(\Omega, \mathbb{R})$ be an initial condition. The orbit $\mathcal{O}(\xi_0)$ is *stable in probability* provided that, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exist a $\delta > 0$

and $\alpha \in (0, 1)$ such that $P(\mathbf{N}_{\xi-\xi_0} < \delta) > 1 - \alpha$ implies that $P(\mathbf{N}_{f^n(\xi)-f^n(\xi_0)} < \varepsilon) > 1 - \lambda$ for all $n \in \mathbb{N}$.

According to Definition 3.2, given an initial condition $\xi_0 \in L_0(\Omega, \mathbb{R})$, we say that the orbit $\mathcal{O}(\xi_0)$ is *unstable in probability* provided that, there exist an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that for every $\delta > 0$ and $\alpha \in (0, 1)$ we can find a $\xi \in L_0(\Omega, \mathbb{R})$ with the following properties:

- (i) $P(\mathbf{N}_{\xi-\xi_0} < \delta) > 1 - \alpha$
- (ii) There exists an $N \in \mathbb{N}$ such that $P(\mathbf{N}_{f^N(\xi)-f^N(\xi_0)} < \varepsilon) \leq 1 - \lambda$.

Example 3.5. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ be defined by $f(\xi) = -\xi$, and take an arbitrary ξ_0 . Now it is enough to choose $\delta = \varepsilon$ and $\alpha = \lambda$ to see that $\mathcal{O}(\xi_0)$ is stable in probability.

Now we will consider the attractivity conditions of fixed points.

Definition 3.3. Let ξ^* be a fixed point of $f : U \rightarrow U$, where U is an open subset of $L_0(\Omega, \mathbb{R})$. Then we say that ξ^* is a *sink in probability* provided that, ξ^* is stable in probability and, there exist an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $P - \lim \mathbf{N}_{f^n(\xi)-\xi^*} = \Theta$ whenever $P(\mathbf{N}_{\xi-\xi^*} < \varepsilon) > 1 - \lambda$.

Before going further, it will be convenient to introduce another concept, following [21].

Definition 3.4. A mapping $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ is *random K -Lipschitz* if there exists a $K \in L_0^+(\Omega)$ such that $\mathbf{N}_{f(\xi)-f(\eta)}(\omega) \leq K(\omega) \mathbf{N}_{\xi-\eta}(\omega)$ a.s. for every $\xi, \eta \in L_0(\Omega, \mathbb{R})$.

Theorem 3.6. Let ξ^* be a fixed point of $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$. If f is random K -Lipschitz with $K(\omega) < 1$ a.s., then ξ^* is a sink in probability.

Proof. Let $\xi \in L_0(\Omega, \mathbb{R})$. Then we have

$$\mathbf{N}_{f(\xi)-\xi^*}(\omega) \leq K(\omega) \mathbf{N}_{\xi-\xi^*}(\omega) \text{ a.s.}$$

After n steps we get

$$\mathbf{N}_{f^n(\xi)-\xi^*}(\omega) \leq K^n(\omega) \mathbf{N}_{\xi-\xi^*}(\omega) \text{ a.s.} \quad (3.1)$$

Now let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given. If we choose $\delta = \varepsilon$ and $\alpha = \lambda$, then $P(\mathbf{N}_{\xi-\xi^*} < \delta) > 1 - \alpha$ implies that $P(\mathbf{N}_{f^n(\xi)-\xi^*} < \varepsilon) > 1 - \lambda$ for all $n \in \mathbb{N}$. Thus ξ^* is stable in probability. Now letting $n \rightarrow \infty$ in (3.1) completes the proof. $\square$

Now we are ready for

Example 3.7. Consider the function $T : \mathbb{R} \rightarrow \mathbb{R}$, $T(x) = ax$ where a is constant and $|a| < 1$ (see [18]). Now let $(\Omega, \mathcal{A}, P)$ be a probability space and $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ be defined by $f(\xi) = a\xi$. Then the fixed point $\xi^* = \Theta$ is a sink in probability if we take $K(\omega) = |a|$ a.s.

Now we will consider the attracting subsets of $L_0(\Omega, \mathbb{R})$.

Definition 3.5. Let $f : U \rightarrow U$, where U is an open subset of $L_0(\Omega, \mathbb{R})$ and A be a compact subset of U . Then A is *attractor in probability* if $f(A) = A$, there exist an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that given $\xi_0 \in U$ with $P(X_{\xi_0 A} < \varepsilon) > 1 - \lambda$ implies that $P - \lim X_{f^n(\xi_0)A} = \Theta$. If A is attractor in probability, then the set of all initial conditions ξ_0 with $P - \lim X_{f^n(\xi_0)A} = \Theta$ will be called the *basin of attraction of A* .

Now let us reconsider Example 3.7. Let $A = \{\Theta\}$, $\varepsilon = 1$, $\lambda \in (0, 1)$. Then, any $\xi_0 \in L_0(\Omega, \mathbb{R})$ with $P(X_{\xi_0 A} < 1) > 1 - \lambda$ implies that

$$P - \lim X_{f^n(\xi_0)A} = \Theta.$$

Hence A is an attractor in probability.

Now we introduce another

Definition 3.6. Let ξ^* be a fixed point of $f : U \rightarrow U$, where U is an open subset of $L_0(\Omega, \mathbb{R})$. Then we say that ξ^* is a *source (repeller) in probability* provided that, there exist an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that whenever $\xi^* \neq \xi$ and $P(\mathbf{N}_{\xi - \xi^*} < \varepsilon) > 1 - \lambda$ we have $P(\mathbf{N}_{f^N(\xi) - \xi^*} < \varepsilon) \leq 1 - \lambda$ for some $N \in \mathbb{N}$.

For instance, the fixed point considered in Example 3.4 is a source in probability. To see this, let $\varepsilon = 1$ and $\lambda = \frac{1}{2}$. If $|\xi| < 1$, then we can find a large enough $N \in \mathbb{N}$ such that $2^N |\xi| \geq 1$ and hence we get $P(2^N |\xi| < 1) < \frac{1}{2}$ whenever $P(|\xi| < 1) > \frac{1}{2}$.

Now we introduce the probabilistic counterpart of a concept which yields an important area of research in topological dynamical systems.

Definition 3.7. Let $B \subset L_0(\Omega, \mathbb{R})$. A homeomorphism $f : B \rightarrow B$ is *expansive in probability* if there exist an $\varepsilon_0 > 0$ such that for every $\xi, \eta \in B$ with $\xi \neq \eta$ we can find a $\lambda \in (0, 1)$ and an $N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that

$$P(\mathbf{N}_{f^N(\xi) - f^N(\eta)} < \varepsilon_0) \leq 1 - \lambda.$$

Here the number ε_0 will be called the *expansiveness constant*.

Example 3.8. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$ be defined by $f(\xi) = 3\xi$. Then $f^N(\xi) = 3^N \xi$ for all $N \in \mathbb{N}_0$. Now let $\varepsilon_0 > 0$ be fixed and arbitrary, and $\xi \neq \eta$. Then, there exists an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $P(\mathbf{N}_{\xi - \eta} < \varepsilon) \leq 1 - \lambda$. Now if $\varepsilon_0 < \varepsilon$, then we have $\frac{\varepsilon_0}{3^N} < \varepsilon$ for all $N \in \mathbb{N}_0$. Thus we get

$$1 - \lambda \geq P(\mathbf{N}_{\xi - \eta} < \varepsilon) \geq P\left(\mathbf{N}_{\xi - \eta} < \frac{\varepsilon_0}{3^N}\right).$$

Similarly, if $\varepsilon_0 \geq \varepsilon$, we can find a large enough $N \in \mathbb{N}_0$ such that $\frac{\varepsilon_0}{3^N} < \varepsilon$. Thus we get

$$1 - \lambda \geq P(\mathbf{N}_{\xi - \eta} < \varepsilon) \geq P\left(\mathbf{N}_{\xi - \eta} < \frac{\varepsilon_0}{3^N}\right).$$

Hence f is expansive in probability with an expansiveness constant ε_0 .

Now let us consider a related

Definition 3.8. Let $f : A \rightarrow A$ be a homeomorphism, A be a compact subset of $L_0(\Omega, \mathbb{R})$ and $\xi, \eta \in A$. Then we say that ξ and η are *proximal in probability* if there is a sequence (n_k) of $\mathbb{Z}$ such that $P - \lim \mathbf{N}_{f^{n_k}(\xi) - f^{n_k}(\eta)} = \Theta$ as $k \rightarrow \infty$. We

also say that ξ and η with $\xi \neq \eta$ are *distal in probability* if there exist an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $P(\mathbf{N}_{f^m(\xi)-f^m(\eta)} < \varepsilon) \leq 1 - \lambda$ for all $m \in \mathbb{Z}$. If every distinct $\xi, \eta \in A$ are distal in probability, then we say that f is *distal in probability*.

Now, before giving a result related to the concept of distal in probability, we define that a homeomorphism $f : A \rightarrow A$ of a compact set $A \subset L_0(\Omega, \mathbb{R})$ is *uniformly equicontinuous* if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exist a $\delta > 0$ and $\alpha \in (0, 1)$ such that whenever $\xi, \eta \in A$ with $P(\mathbf{N}_{\xi-\eta} < \delta) > 1 - \alpha$ we have

$$P(\mathbf{N}_{f^m(\xi)-f^m(\eta)} < \varepsilon) > 1 - \lambda$$

for all $m \in \mathbb{Z}$.

Theorem 3.9. *Let $A \subset L_0(\Omega, \mathbb{R})$ be compact and $f : A \rightarrow A$ be a uniformly equicontinuous homeomorphism. Then f is distal in probability.*

Proof. Assume that f is not distal in probability. Then there exist $\xi, \eta \in A$ which are proximal in probability, and hence $P\text{-}\lim_{f^{n_k}(\xi)-f^{n_k}(\eta)} \mathbf{N} = \Theta$ for $(n_k) \subset \mathbb{Z}$. Now let us denote $f^{n_k}(\xi) = \xi_k$ and $f^{n_k}(\eta) = \eta_k$. Let $\xi \neq \eta$. Then there exist an $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $P(\mathbf{N}_{\xi-\eta} < \varepsilon) \leq 1 - \lambda$. Moreover, for any $\delta > 0$ and $\alpha \in (0, 1)$ there exists a $k \in \mathbb{N}$ such that $P(\mathbf{N}_{\xi_k-\eta_k} < \delta) > 1 - \alpha$ but $P(\mathbf{N}_{f^{-n_k}(\xi_k)-f^{-n_k}(\eta_k)} < \varepsilon) \leq 1 - \lambda$. Thus f is not uniformly equicontinuous. $\square$

In the rest of the paper, we will study on homeomorphisms which are expansive in probability. First, we extend a result in [30] given for metric spaces, to the setting of random normed space of random variables.

Theorem 3.10. *Let $f : A \rightarrow A$ be a homeomorphism that is expansive in probability, where $A \subset L_0(\Omega, \mathbb{R})$ is compact. Then f^n is also expansive in probability on A for any $n \in \mathbb{N}$.*

Proof. Let $\xi \neq \eta$, where $\xi, \eta \in A$. Since f is expansive in probability, there exist an $\varepsilon_0 > 0$ such that we can find a $\lambda \in (0, 1)$ and an $N \in \mathbb{N}_0$ with

$$P(\mathbf{N}_{f^N(\xi)-f^N(\eta)} < \varepsilon_0) \leq 1 - \lambda.$$

Now let $f^i : A \rightarrow A$ be a homeomorphism for each $i = 1, \dots, n$. Since $L_0(\Omega, \mathbb{R})$ is a uniform space and A is compact, each f^i is uniformly continuous. Hence for each i , there exist a $\delta_i > 0$ and $\alpha_i \in (0, 1)$ such that $P(\mathbf{N}_{\xi-\eta} < \varepsilon_0) \leq 1 - \lambda$ implies that $P(\mathbf{N}_{f^i(\xi)-f^i(\eta)} < \delta_i) \leq 1 - \alpha_i$. Now define $f^n = \varphi$. Then there exists a $k \in \mathbb{N}$ such that $i + N = kn$ with $f^i(f^N(\xi)) = \varphi^k(\xi)$ for some $i \in \{1, \dots, n\}$ and for every $\xi \in A$. Now set $\min\{\delta_i : 1 \leq i \leq n\} = \delta$ and $\min\{\alpha_i : 1 \leq i \leq n\} = \alpha$. Hence we have

$$1 - \alpha \geq 1 - \alpha_i \geq P(\mathbf{N}_{\varphi^k(\xi)-\varphi^k(\eta)} < \delta_i) \geq P(\mathbf{N}_{\varphi^k(\xi)-\varphi^k(\eta)} < \delta),$$

which shows that f^n is expansive in probability with an expansiveness constant δ . $\square$

Before proceeding, we will need a

Definition 3.9. Let $f : L_0(\Omega, \mathbb{R}) \rightarrow L_0(\Omega, \mathbb{R})$. Then $\xi, \eta \in L_0(\Omega, \mathbb{R})$ are said to be *asymptotic in probability* under f , provided that $P\text{-}\lim_{f^n(\xi)-f^n(\eta)} \mathbf{N} = \Theta$.

Finally, we will extend some of the results in [5] and [6] given for metric spaces, to the setting of random variables.

Theorem 3.11. *Let $f : A \rightarrow A$ be an expansive homeomorphism, where A is an infinite compact subset of $L_0(\Omega, \mathbb{R})$ and $\xi \neq \eta$, where $\xi, \eta \in A$. Moreover, let ε_0 be the expansiveness constant for f . If there is an $N \in \mathbb{N}$ such that $n > N$ implies $\mathbf{N}_{f^n(\xi)-f^n(\eta)} < \varepsilon_0$ a.s., then ξ and η are asymptotic in probability.*

Proof. Assume on the contrary that ξ and η are not asymptotic in probability. Then there exist a $\delta > 0$ and $\alpha \in (0, 1)$, and an increasing sequence (n_k) of positive integers such that

$$P(\mathbf{N}_{f^{n_k}(\xi)-f^{n_k}(\eta)} < \delta) \leq 1 - \alpha$$

with $\lim_{k \rightarrow \infty} f^{n_k}(\xi) = \xi_0$ and $\lim_{k \rightarrow \infty} f^{n_k}(\eta) = \eta_0$. Then we have $\xi_0 \neq \eta_0$. Now let $m \in \mathbb{N}$. Then, for all k sufficiently large, we may set $n_k + m > N$. Hence we have

$$P(\mathbf{N}_{f^{n_k+m}(\xi)-f^{n_k+m}(\eta)} < \varepsilon_0) = 1$$

by hypothesis. Since $\lim_{k \rightarrow \infty} f^{n_k+m}(\xi) = f^m(\xi_0)$ and $\lim_{k \rightarrow \infty} f^{n_k+m}(\eta) = f^m(\eta_0)$, we get

$$P(\mathbf{N}_{f^m(\xi_0)-f^m(\eta_0)} \leq \varepsilon_0) = 1$$

for all $m \in \mathbb{N}$, but this contradicts the fact that f is expansive in probability. To see this, let

$$f^m(\xi_0) = p, \quad f^{n_k+m}(\xi) = q, \quad f^{n_k+m}(\eta) = r \quad \text{and} \quad f^m(\eta_0) = s.$$

Now let

$$\begin{aligned} M_1 &= \left\{ \omega \in \Omega : \mathbf{N}_{p-q}(\omega) < \frac{\beta}{2} \right\}, \\ M_2 &= \{ \omega \in \Omega : \mathbf{N}_{q-r}(\omega) < \varepsilon_0 \}, \\ M_3 &= \left\{ \omega \in \Omega : \mathbf{N}_{p-r}(\omega) < \varepsilon_0 + \frac{\beta}{2} \right\}, \end{aligned}$$

where $\beta > 0$ is arbitrary. Then we have

$$\begin{aligned} P(M_3) &\geq P(M_1 \cap M_2) \geq P(M_1) + P(M_2) - 1 \\ &> \left(1 - \frac{\gamma}{2}\right) + 1 - 1 = 1 - \frac{\gamma}{2} \end{aligned}$$

for sufficiently large k , where $\gamma \in (0, 1)$ is arbitrary. Now let

$$\begin{aligned} M_4 &= \{ \omega \in \Omega : \mathbf{N}_{p-s}(\omega) < \varepsilon_0 + \beta \}, \\ M_5 &= \left\{ \omega \in \Omega : \mathbf{N}_{r-s}(\omega) < \frac{\beta}{2} \right\}. \end{aligned}$$

Then we have

$$\begin{aligned} P(M_4) &\geq P(M_3 \cap M_5) \geq P(M_3) + P(M_5) - 1 \\ &> \left(1 - \frac{\gamma}{2}\right) + \left(1 - \frac{\gamma}{2}\right) - 1 = 1 - \gamma \end{aligned}$$

for sufficiently large k . This implies that $P(M_4) = 1$ and hence $P(\mathbf{N}_{p-s} \leq \varepsilon_0) = 1$. $\square$

Theorem 3.12. *Let $g : L_0(\Omega, \mathbb{R}) \longrightarrow L_0(\Omega, \mathbb{R})$ be a homeomorphism such that g^{-1} is uniformly continuous and $f : L_0(\Omega, \mathbb{R}) \longrightarrow L_0(\Omega, \mathbb{R})$ be a homeomorphism which is expansive in probability. Then gfg^{-1} is a homeomorphism which is expansive in probability.*

Proof. Let the expansiveness constant of f be ε_0 . If $\xi_1 \neq \xi_2$, then $g^{-1}(\xi_1) \neq g^{-1}(\xi_2)$ and hence there exist a $\lambda \in (0, 1)$ and an $N \in \mathbb{N}_0$ such that

$$P\left(\mathbf{N}_{f^N g^{-1}(\xi_1) - f^N g^{-1}(\xi_2)} < \varepsilon_0\right) \leq 1 - \lambda.$$

Moreover, since g^{-1} is uniformly continuous, there exist a $\delta_0 > 0$ and $\alpha \in (0, 1)$ such that $P(\mathbf{N}_{g(\xi) - g(\eta)} < \delta_0) > 1 - \alpha$ implies that $P(\mathbf{N}_{\xi - \eta} < \varepsilon_0) > 1 - \lambda$. Hence we get $P(\mathbf{N}_{g f^N g^{-1}(\xi_1) - g f^N g^{-1}(\xi_2)} < \delta_0) \leq 1 - \alpha$, which completes the proof. $\square$

Conclusion

In the current paper, we have investigated the concept of iteration of functions of random variables via the theory of random normed (RN) spaces. In this context, we have introduced some stochastic counterparts of some of the well known concepts of classical topological dynamics, and obtained certain results. A further study can be the chaotic dynamics of random variables. This will be our future work.

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Conflict of Interest

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