

CERTAIN SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH q -CALCULUS

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ABSTRACT. The purpose of the present paper is to investigate a new subclass of harmonic univalent functions associated with q -calculus. Extreme points, distortion bounds, convolution theorem and convex combination are determined for this class. Relevant Connections of the results presented herewith various known results are briefly indicated.

1. INTRODUCTION

A continuous complex valued function $f = u + iv$ is said to be harmonic in a simply-connected domain $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply-connected region, we represent $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. The analytic and co-analytic part of the function f are h and g , respectively. The necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is $|h'(z)| > |g'(z)|$, $z \in \mathbb{D}$.

In 1984, Clunie and Sheil-Small [5], Duren [8], (see also [1, 16]) investigated the class S_H of function $f = h + \bar{g}$ which are harmonic univalent and sense-preserving in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_{\bar{z}}(0) - 1 = 0$. They investigated several interesting properties of harmonic univalent functions and opens up a new direction of research in geometric function theory. After appearance of this paper Silverman [20] investigated a subclass of harmonic univalent functions with negative coefficients. Further these results were generalized and improved by Silverman and Silvia [21], Jahangiri [11]. After the investigation of these subclasses the study of harmonic univalent and multivalent functions became an active area of research in geometric function theory.

In fact, several authors such as [2, 7, 12, 14] presented a systematic and unified study of harmonic univalent functions. Recently, application of quantum calculus and post quantum calculus on analytic and harmonic univalent functions has a new area of research in this field. Various subclasses of analytic and harmonic univalent functions involving q -calculus were recently investigated and studied by eminent researchers of this field. They obtain several new and interesting results. Further, several authors studied bi-univalent functions and Hankel determinant of univalent functions associated with q -calculus. Noteworthy contribution in this direction may be found in [2, 12, 17, 18]. Recently, in 2020, Srivastava [22] had a detailed study of q -calculus and fractional q -calculus and its applications in his survey -cum-expository review article.

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Now, we recall the definition of q -calculus which was first introduced by Jackson [9, 10], (see also [3]). The q -number for $k \in \mathbb{N}$ is defined in the following way

$$[k]_q = \frac{1 - q^k}{1 - q}, 0 \leq q < 1.$$

$$\implies [k]_q = \sum_{i=0}^{k-1} q^i.$$

$$\lim_{k \rightarrow \infty} [k]_q = \frac{1}{1 - q}$$

and

$$\lim_{q \rightarrow 1} [k]_q = k.$$

The q -derivative for a function f is defined by

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q - 1)z}, q \neq 1, z \neq 0$$

and $D_q(f(0)) = f'(0)$, provided $f'(0)$ exists.

The q -Jackson definite integral of the function f is defined by

$$\int_0^z f(t) d_q t = (1 - q)z \sum_{n=0}^{\infty} f(zq^n) q^n, z \in \mathbb{C}.$$

A complex-valued harmonic univalent and sense-preserving mapping f defined on open unit disk $\mathbb{U}$ can be written as

$$f(z) = h(z) + \overline{g(z)} \quad (1.1)$$

where $h(z) = z + \sum_{k=j}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$, $|b_1| < 1$ are analytic in $\mathbb{U}$.

Let $S_H(n, \alpha, \beta, q, j)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the following criteria

$$\sum_{k=j}^{\infty} k^n \{k[k]_q(1 + \beta) - (1 - \beta + 2\alpha\beta)\} |a_k| + \sum_{k=2}^{\infty} k^n \{k[k]_q(1 + \beta) - (1 - \beta + 2\alpha\beta)\} |b_k| \leq 2\beta(1 - \alpha)(1 - |b_1|), \quad (1.2)$$

where $n \in \mathbb{N}_0, j \geq 2, 0 < \beta \leq 1, 0 \leq \alpha < 1$.

Further, $S_H^0(n, \alpha, \beta, q, j)$ denotes the subclass of $S_H(n, \alpha, \beta, q, j)$ for which $b_1 = 0$.

By specializing the parameter we obtain the following known subclasses studied earlier by various researchers which are specified below.

1. If we let $q \rightarrow 0$ and $j = 2$ in the class $S_H(n, \alpha, \beta, q, j)$ then it reduces to the class $S_H(n, \alpha, \beta)$ studied by Pathak et al. [15].
2. If we let $q \rightarrow 0, j = 2, n = 0, \beta = 1$ in the class $S_H(n, \alpha, \beta, q, j)$ then it reduces to the class $HS(\alpha)$ studied by Öztürk and Yalcin [13].
3. If we let $q \rightarrow 0, j = 2, n = 1, \beta = 1$ in the class $S_H(n, \alpha, \beta, q, j)$ then it reduces to the class $HC(\alpha)$ studied by Öztürk and Yalcin [13], see also [6, 19].
4. If we let $q \rightarrow 1, j = 2, \beta = 1, \alpha = 0, n = 0$ and $n = 1$ in the class $S_H(n, \alpha, \beta, q, j)$ then it reduces to the classes HS and HC studied by Avci and Zlotkiewicz [4].

2. MAIN RESULTS

First, we prove that the class $S_H(n, \alpha, \beta, q, j)$ is univalent and sense-preserving in $\mathbb{U}$.

Theorem 2.1. *The function $f \in S_H(n, \alpha, \beta, q, j)$ is harmonic univalent and sense-preserving in $\mathbb{U}$.*

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\ &= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{z_1 - z_2 + \sum_{k=j}^{\infty} a_k(z_1^k - z_2^k)} \right| \\ &> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=j}^{\infty} k|a_k|} \\ &\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{k^n \{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\} |b_k|}{2\beta(1-\alpha)}}{1 - \sum_{k=j}^{\infty} \frac{k^n \{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\} |a_k|}{2\beta(1-\alpha)}} \geq 0. \end{aligned}$$

Thus f is univalent in $\mathbb{U}$.

Next, we prove that f is sense-preserving in $\mathbb{U}$. This is because

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=j}^{\infty} k|a_k|z^k > 1 - \sum_{k=j}^{\infty} k|a_k| \\ &\geq 1 - \sum_{k=j}^{\infty} \frac{k^n \{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\} |a_k|}{2\beta(1-\alpha)} \\ &\geq \sum_{k=1}^{\infty} \frac{k^n \{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\} |b_k|}{2\beta(1-\alpha)} \\ &\geq \sum_{k=1}^{\infty} k|b_k| > \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|. \end{aligned}$$

□

In our next theorem we obtain bounds for $f \in S_H(n, \alpha, \beta, q, j)$.

Theorem 2.2. *If $f \in S_H(n, \alpha, \beta, q, j)$ then*

$$|f(z)| \leq (1 + |b_1||z| + |b_2||z|^2 + \dots + |b_{j-1}||z|^{j-1}) + \frac{2\beta(1-\alpha)}{j^n[j]_q(1+\beta) - (1-\beta+2\alpha\beta)} \left[1 - \sum_{k=1}^{j-1} \frac{k^n \{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\} |b_k|}{2\beta(1-\alpha)} \right]$$

and

$$|f(z)| \geq (1 - |b_1||z| - |b_2||z|^2 - \dots - |b_{j-1}||z|^{j-1}) - \frac{2\beta(1-\alpha)}{j^n[j]_q(1+\beta) - (1-\beta+2\alpha\beta)} \left[1 - \sum_{k=1}^{j-1} \frac{k^n \{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\} |b_k|}{2\beta(1-\alpha)} \right].$$

Proof. We only prove the right-hand inequality. The proof for the left hand inequality is similar and will be omitted.

Let $f \in S_H(n, \alpha, \beta, q, j)$

$$\begin{aligned} |f(z)| &\leq (1 + |b_1||z| + |b_2||z|^2 + \dots + |b_{j-1}||z|^{j-1}) + \sum_{k=j}^{\infty} (|a_k| + |b_k|)|z|^k \\ &= (1 + |b_1||z| + |b_2||z|^2 + \dots + |b_{j-1}||z|^{j-1}) \end{aligned}$$

$$\begin{aligned}
& + \frac{2\beta(1-\alpha)}{j^n[j[j]_q(1+\beta) - (1-\beta+2\alpha\beta)]} |z|^j \sum_{k=j}^{\infty} \frac{j^n\{j[j]_q(1+\beta) - (1-\beta+2\alpha\beta)\}}{2\beta(1-\alpha)} (|a_k| + |b_k|) \\
& \leq (1 + |b_1||z| + |b_2||z|^2 + \dots + |b_{j-1}||z|^{j-1}) + \\
& + \frac{2\beta(1-\alpha)}{j^n[j[j]_q(1+\beta) - (1-\beta+2\alpha\beta)]} |z|^j \sum_{k=j}^{\infty} \frac{k^n\{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\}}{2\beta(1-\alpha)} (|a_k| + |b_k|) \\
& = (1 + |b_1||z| + |b_2||z|^2 + \dots + |b_{j-1}||z|^{j-1}) + \\
& \frac{2\beta(1-\alpha)}{j^n[j[j]_q(1+\beta) - (1-\beta+2\alpha\beta)]} \left[1 - \sum_{k=1}^{j-1} \frac{k^n\{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\}}{2\beta(1-\alpha)} |b_k| \right].
\end{aligned}$$

□

Theorem 2.3. *The extreme points of $S_H^0(n, \alpha, \beta, q, j)$ are only the functions of the form $z + a_k z^k$ or $z + \overline{b_t z^t}$ with*

$$\begin{aligned}
|a_k| &= \frac{2\beta(1-\alpha)}{k^n[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}, k = j, j+1, j+2, \dots \\
|b_t| &= \frac{2\beta(1-\alpha)}{t^n[t[t]_q(1+\beta) - (1-\beta+2\alpha\beta)]}, t = 2, 3, \dots
\end{aligned}$$

Proof. Let

$$f(z) = z + \sum_{k=j}^{\infty} a_k z^k + \sum_{k=2}^{\infty} \overline{b_k z^k}$$

is such that

$$\sum_{k=j}^{\infty} \frac{k^n\{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\}}{2\beta(1-\alpha)} |a_k| + \sum_{k=2}^{\infty} \frac{k^n\{k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)\}}{2\beta(1-\alpha)} |b_k| < 1, \quad (2.1)$$

$$a_k > 0$$

Then if $\mu > 0$ is small enough we can replace a_k by $a_k - \mu$, $a_k + \mu$ and we obtain two functions that satisfy the condition (2.1) for which one obtains

$$f(z) = \frac{1}{2}[f_1(z) + f_2(z)].$$

Thus, we say that f is not a possible extreme point of $S_H^0(n, \alpha, \beta, q, j)$.

Further, suppose that $f \in S_H^0(n, \alpha, \beta, q, j)$ be such that

$$\sum_{k=j}^{\infty} \frac{k^n[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |a_k| + \sum_{k=2}^{\infty} \frac{k^n[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |b_k| = 1, \quad (2.2)$$

$$a_k \neq 0, b_t \neq 0.$$

If $\mu > 0$ is small enough and if δ, ε with $|\delta| = |\varepsilon| = 1$ are properly chosen complex numbers, then leaving all but a_k and b_t coefficients of $f(z)$ unchanged and replacing a_k, b_t by

$$a_k + \mu \frac{2\beta(1-\alpha)}{k^n[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]} \delta; b_t - \mu \frac{2\beta(1-\alpha)}{t^n[t[t]_q(1+\beta) - (1-\beta+2\alpha\beta)]} \varepsilon,$$

$$a_k - \mu \frac{2\beta(1-\alpha)}{k^n[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]} \delta; b_t + \mu \frac{2\beta(1-\alpha)}{t^n[t[t]_q(1+\beta) - (1-\beta+2\alpha\beta)]} \varepsilon,$$

we obtain functions $f_1(z), f_2(z)$ that satisfy condition (2.2) such that

$$f(z) = \frac{1}{2}[f_1(z) + f_2(z)].$$

In this case f cannot be an extreme point. Thus for $|a_k| = \frac{2\beta(1-\alpha)}{k^n[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}$, $|b_t| = \frac{2\beta(1-\alpha)}{t^n[t[t]_q(1+\beta) - (1-\beta+2\alpha\beta)]}$, $f(z) = z + a_k z^k$ or $f(z) = z + \overline{b_t} z^t$ are extreme points of $S_H^0(n, \alpha, \beta, q, j)$. $\square$

To prove our next theorem we define the convolution of two harmonic functions.

Definition 2.1. For harmonic functions $f(z)$ and $F(z)$ of the form defined by

$$f(z) = z + \sum_{k=j}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k} z^k \quad (2.3)$$

and

$$F(z) = z + \sum_{k=j}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k} z^k. \quad (2.4)$$

Then the convolution of $f(z)$ and $F(z)$ is denoted by $f(z) * F(z)$ and defined by

$$(f * F)(z) = f(z) * F(z)$$

$$(f * F)(z) = z + \sum_{k=j}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} \overline{b_k B_k} z^k. \quad (2.5)$$

Using this definition we show that the family $S_H(n, \alpha, \beta, q, j)$ is closed under convolution.

Theorem 2.4. For $0 \leq \alpha < 1, 0 < \beta \leq 1, 0 < q < 1, j \geq 2$ let $f \in S_H(n, \alpha, \beta, q, j)$ and $F \in S_H(m, \alpha, \beta, q, j)$. Then $(f * F)(z) \in S_H(m+n, \alpha, \beta, q, j) \subseteq S_H(n, \alpha, \beta, q, j)$.

Proof. Let $f(z)$ and $F(z)$ defined by (2.3) and (2.4) belong to the classes $S_H(n, \alpha, \beta, q, j)$ and $S_H(m, \alpha, \beta, q, j)$ respectively, then the convolution $(f * F)(z)$ is defined by (2.5). For $F \in S_H(m, \alpha, \beta, q, j)$ we note that

$$\begin{aligned} & \sum_{k=j}^{\infty} \frac{k^m[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |A_k| \\ & + \sum_{k=1}^{\infty} \frac{k^m[k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |B_k| \leq 1. \end{aligned}$$

This suggests that $k^m|A_k| \leq 1$ and $k^m|B_k| \leq 1$

Now, for the convolution function $(f * F)(z)$ we have

$$\begin{aligned} & \sum_{k=j}^{\infty} \frac{k^{m+n} [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} (|a_k A_k|) \\ & + \sum_{k=1}^{\infty} \frac{k^{m+n} [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} (|b_k B_k|) \\ & \leq \sum_{k=j}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |a_k| \\ & + \sum_{k=1}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |b_k| \leq 1. \end{aligned}$$

Thus, $(f * F)(z) \in S_H(m+n, \alpha, \beta, q, j)$. □

Theorem 2.5. *The class $S_H(m, \alpha, \beta, q, j)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \dots$ let $f_i(z) \in S_H(n, \alpha, \beta, q, j)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=j}^{\infty} a_{k_i} z^k + \sum_{k=1}^{\infty} \overline{b_{k_i}} z^k.$$

Then by definition

$$\begin{aligned} & \sum_{k=j}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |a_{k_i}| \\ & + \sum_{k=1}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |b_{k_i}| \leq 1. \end{aligned}$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=j}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) z^k + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} t_i \overline{b_{k_i}} z^k.$$

Now,

$$\begin{aligned} & \sum_{k=j}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}| \right) \\ & + \sum_{k=1}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} \left(\sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \\ & = \sum_{i=1}^{\infty} t_i \left[\sum_{k=j}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |a_{k_i}| \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \frac{k^n [k[k]_q(1+\beta) - (1-\beta+2\alpha\beta)]}{2\beta(1-\alpha)} |b_{k_i}| \right] \\ & \leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Thus, $\sum_{i=1}^{\infty} t_i f_i(z) \in S_H(m, \alpha, \beta, q, j)$ and the proof is complete. □

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