

ON ALMOST (mI, nJ) -CONTINUOUS MULTIFUNCTIONS

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ABSTRACT. Let $mIO(X)$ be the family of $\star$ -open (resp. α - I -open, pre- I -open, semi- I -open, β - I -open, etc.) sets in an ideal topological space (X, τ, I) . By using $mIO(X)$, we introduce and investigate the notion of an almost (mI, nJ) -continuous multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$. As a special case of almost (mI, nJ) -continuous multifunctions, we obtain the notion of an almost m - I -continuous multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma)$.

1. INTRODUCTION

The notions of minimal structures, m -spaces, m -continuity and M -continuity are introduced and investigated in [9] and [14]. By using these notions, the present authors obtained some unified theory of continuity for multifunctions [11], [15].

The notion of ideal topological spaces was initiated in [8], [18]. As generalizations of open sets in an ideal topological space, several authors began to study the notions of semi- I -open sets, pre- I -open sets, α - I -open sets, b - I -open sets and β - I -open sets, etc. In this paper, let $mIO(X)$ be the family of $\star$ -open (resp. α - I -open, pre- I -open, semi- I -open, β - I -open, etc.) sets in an ideal topological space (X, τ, I) . By using $mIO(X)$, in [13] the present authors introduced the notion of an (mI, nJ) -continuous multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$. In this paper, we introduce and obtain some characterizations of almost (mI, nJ) -continuous multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$.

Throughout the present paper, spaces (X, τ) and (Y, σ) always mean topological spaces and $F : (X, \tau) \rightarrow (Y, \sigma)$ presents a multifunction. For a multifunction $F : X \rightarrow Y$, we shall denote the upper and lower inverses of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Let $\mathcal{P}(Y)$ be the collection of all nonempty subsets of Y . For any open set V of Y , we denote $V^+ = \{B \in \mathcal{P}(Y) : B \subset V\}$ and $V^- = \{B \in \mathcal{P}(Y) : B \cap V \neq \emptyset\}$ [17].

2. PRELIMINARIES

Definition 2.1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X [14] if $\emptyset \in m$ and $X \in m$.

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Each member of m is said to be *m-open* and the complement of an *m-open* set is said to be *m-closed*. A set X with an *m-structure* m is called an *m-space* and is denoted by (X, m) .

Definition 2.2. Let X be a nonempty set and m an *m-structure* on X . For a subset A of X , the *m-closure* of A and the *m-interior* of A are defined in [9] as follows:

- (1) $\text{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m\},$
- (2) $\text{mInt}(A) = \cup\{U : U \subset A, U \in m\}.$

Lemma 2.1. [9] Let (X, m) be an *m-space*. For subsets A and B of X , the following properties hold:

- (1) $\text{mCl}(X \setminus A) = X \setminus \text{mInt}(A)$ and $\text{mInt}(X \setminus A) = X \setminus \text{mCl}(A),$
- (2) If $(X \setminus A) \in m$, then $\text{mCl}(A) = A$ and if $A \in m$, then $\text{mInt}(A) = A,$
- (3) $\text{mCl}(\emptyset) = \emptyset, \text{mCl}(X) = X, \text{mInt}(\emptyset) = \emptyset$ and $\text{mInt}(X) = X,$
- (4) If $A \subset B$, then $\text{mCl}(A) \subset \text{mCl}(B)$ and $\text{mInt}(A) \subset \text{mInt}(B),$
- (5) $A \subset \text{mCl}(A)$ and $\text{mInt}(A) \subset A,$
- (6) $\text{mCl}(\text{mCl}(A)) = \text{mCl}(A)$ and $\text{mInt}(\text{mInt}(A)) = \text{mInt}(A).$

Definition 2.3. An *m-structure* m on a nonempty set X is said to have *property B* [9] if the union of any family of subsets belonging to m belongs to m .

Lemma 2.2. [16] Let X be a nonempty set and m an *m-structure* with *property B*. Then, the following properties hold:

- (1) $\text{mInt}(A) = A$ if and only if $A \in m,$
- (2) $\text{mCl}(A) = A$ if and only if A is *m-closed*,
- (3) $\text{mInt}(A) \in m$ and $\text{mCl}(A)$ is *m-closed*.

3. IDEAL TOPOLOGICAL SPACES

Let (X, τ) be a topological space. The notion of ideals has been introduced in [8] and [18] and further investigated in [7].

Definition 3.1. A nonempty collection I of subsets of a set X is called an *ideal* on X if it satisfies the following two conditions:

- (1) $A \in I$ and $B \subset A$ implies $B \in I,$
- (2) $A \in I$ and $B \in I$ implies $A \cup B \in I.$

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I) . Let (X, τ, I) be an ideal topological space. For any subset A of X , $A^*(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [7]. Hereafter $A^*(I, \tau)$ is simply denoted by A^* . It is well known that $\text{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator on X and the topology generated by Cl^* is denoted by τ^* .

Lemma 3.1. [7] Let (X, τ, I) be an ideal topological space and A, B be subsets of X . Then the following properties hold:

- (1) $A \subset B$ implies $\text{Cl}^*(A) \subset \text{Cl}^*(B),$
- (2) $\text{Cl}^*(X) = X$ and $\text{Cl}^*(\emptyset) = \emptyset,$
- (3) $\text{Cl}^*(A) \cup \text{Cl}^*(B) = \text{Cl}^*(A \cup B).$

Definition 3.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be

- (1) α - I -open [6] if $A \subset \text{Int}(\text{Cl}^*(\text{Int}(A)))$,
- (2) *semi- I -open* [6] if $A \subset \text{Cl}^*(\text{Int}(A))$,
- (3) *pre- I -open* [3] if $A \subset \text{Int}(\text{Cl}^*(A))$,
- (4) *b- I -open* [1] if $A \subset \text{Int}(\text{Cl}^*(A)) \cup \text{Cl}^*(\text{Int}(A))$,
- (5) β - I -open [6] if $A \subset \text{Cl}(\text{Int}(\text{Cl}^*(A)))$,
- (6) *weakly semi- I -open* [4] if $A \subset \text{Cl}^*(\text{Int}(\text{Cl}(A)))$,
- (7) *weakly b- I -open* [10] if $A \subset \text{Cl}(\text{Int}(\text{Cl}^*(A))) \cup \text{Cl}^*(\text{Int}(\text{Cl}(A)))$,
- (8) *strongly β - I -open* [5] if $A \subset \text{Cl}^*(\text{Int}(\text{Cl}^*(A)))$.

Among the sets in Definition 3.2, we have the following relations:

DIAGRAM

$$\begin{array}{ccccccc}
 \text{open} & \Rightarrow & \alpha\text{-}I\text{-open} & \Rightarrow & \text{semi-}I\text{-open} & \Rightarrow & \text{weakly semi-}I\text{-open} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{pre-}I\text{-open} & \Rightarrow & b\text{-}I\text{-open} & \Rightarrow & \text{weakly } b\text{-}I\text{-open} \\
 & & & & \Downarrow & & \Uparrow \\
 & & & & \text{strongly } \beta\text{-}I\text{-open} & \Rightarrow & \beta\text{-}I\text{-open}
 \end{array}$$

The family of all α - I -open (resp. semi- I -open, pre- I -open, b - I -open, β - I -open, weakly semi- I -open, weakly b - I -open, strongly β - I -open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha\text{IO}(X)$ (resp. $\text{SIO}(X)$, $\text{PIO}(X)$, $\text{BIO}(X)$, $\beta\text{IO}(X)$, $\text{WSIO}(X)$, $\text{WBIO}(X)$, $\text{S}\beta\text{IO}(X)$).

Definition 3.3. By $\text{mIO}(X)$, we denote each one of the families τ^* , $\alpha\text{IO}(X)$, $\text{SIO}(X)$, $\text{PIO}(X)$, $\text{BIO}(X)$, $\beta\text{IO}(X)$, $\text{WSIO}(X)$, $\text{WBIO}(X)$, $\text{S}\beta\text{IO}(X)$. Similarly, by $\text{nJO}(Y)$, we denote each one of the families σ^* , $\alpha\text{JO}(Y)$, $\text{SJO}(Y)$, $\text{PJO}(Y)$, $\text{BJO}(Y)$, $\beta\text{JO}(Y)$, $\text{WSJO}(Y)$, $\text{WBJO}(Y)$, $\text{S}\beta\text{JO}(Y)$.

Lemma 3.2. Let (X, τ, I) be an ideal topological space. Then $\text{mIO}(X)$ is an m -structure on X and has property $\mathcal{B}$.

Proof. For example, let $\text{mIO}(X) = \alpha\text{IO}(X)$.

(1) $\emptyset, X \in \alpha\text{IO}(X)$.

(2) Let $A_k \in \alpha\text{IO}(X)$ for each $k \in K$. Then

$$A_k \subset \text{Int}(\text{Cl}^*(\text{Int}(A_k))) \subset \text{Int}(\text{Cl}^*(\text{Int}(\cup_{k \in K} A_k)))$$

and $\cup_{k \in K} A_k \subset \text{Int}(\text{Cl}^*(\text{Int}(\cup_{k \in K} A_k)))$. Hence $\cup_{k \in K} A_k \in \alpha\text{IO}(X)$. $\square$

Definition 3.4. Let (X, τ, I) be an ideal topological space. For a subset A of X , $\text{mCl}_I(A)$ and $\text{mInt}_I(A)$ are defined as follows:

- (1) $\text{mCl}_I(A) = \cap \{F : A \subset F, X \setminus F \in \text{mIO}(X)\}$,
- (2) $\text{mInt}_I(A) = \cup \{U : U \subset A, U \in \text{mIO}(X)\}$.

Let (X, τ, I) be an ideal topological space and $\text{mIO}(X)$ the m -structure on X . If $\text{mIO}(X) = \tau^*$ (resp. $\alpha\text{IO}(X)$, $\text{SIO}(X)$, $\text{PIO}(X)$, $\text{BIO}(X)$, $\beta\text{IO}(X)$, $\text{WSIO}(X)$, $\text{WBIO}(X)$, $\text{S}\beta\text{IO}(X)$), then we have the following:

- (1) $mCl_I(A) = Cl^*(A)$ (resp. $\alpha Cl_I(A)$, $sCl_I(A)$, $pCl_I(A)$, $bCl_I(A)$, $\beta Cl_I(A)$, $wsCl_I(A)$, $wbCl_I(A)$, $s\beta Cl_I(A)$),
 (2) $mInt_I(A) = Int^*(A)$ (resp. $\alpha Int_I(A)$, $sInt_I(A)$, $pInt_I(A)$, $bInt_I(A)$, $\beta Int_I(A)$, $wsInt_I(A)$, $wbInt_I(A)$, $s\beta Int_I(A)$).

4. ALMOST (mI, nJ) -CONTINUOUS MULTIFUNCTIONS

Let n be an m -structure on a set Y , J be an ideal on Y and (Y, n, J) be an ideal topological space. A subset B of Y is said to be nJ -regular-open if $nInt_J(nCl_J(B)) = B$. The complement of an nJ -regular-open set is said to be nJ -regular-closed.

Definition 4.1. A multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is said to be

- (1) *almost (mI, nJ) -continuous* at $x \in X$ if for all nJ -open sets $V_1, V_2 \in nJO(Y)$ such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in mIO(X)$ containing x such that $F(U) \subset nInt_J(nCl_J(V_1))$ and $F(u) \cap nInt_J(nCl_J(V_2)) \neq \emptyset$ for every $u \in U$.
 (2) *almost (mI, nJ) -continuous* if F has this property at each point of X .

Remark. Every (mI, nJ) -continuous multifunction is *almost (mI, nJ) -continuous*. But the converse is not always true. This is shown in Example 7 of [2].

Theorem 4.1. For a multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$, the following properties are equivalent:

- (1) F is almost (mI, nJ) -continuous at $x \in X$;
 (2) for any nJ -regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists a $U \in mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$;
 (3) $x \in mInt_I(F^+(G_1) \cap F^-(G_2))$ for all nJ -regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$;
 (4) $x \in mInt_I(F^+(nInt_J(nCl_J(V_1))) \cap F^-(nInt_J(nCl_J(V_2))))$ for all nJ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$.

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): Let G_1, G_2 be any nJ -regular open sets of Y such that $F(x) \in G_1^+ \cap G_2^-$. Then, there exists $U \in mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Then, we have $x \in U \subset F^+(G_1) \cap F^-(G_2)$. Since $U \in mIO(X)$, we obtain $x \in mInt_I(F^+(G_1) \cap F^-(G_2))$.

(3) $\Rightarrow$ (4): Let V_1, V_2 be any nJ -open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, $F(x) \in (nInt_J(nCl_J(V_1)))^+ \cap (nInt_J(nCl_J(V_2)))^-$. Therefore, by (3) we obtain

$$x \in mInt_I[F^+(nInt_J(nCl_J(V_1))) \cap F^-(nInt_J(nCl_J(V_2)))].$$

(4) $\Rightarrow$ (1): Let V_1, V_2 be any nJ -open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, by (4) $x \in mInt_I[F^+(nInt_J(nCl_J(V_1))) \cap F^-(nInt_J(nCl_J(V_2)))]$. Therefore, there exists $U \in mIO(X)$ containing x such that $x \in U \subset F^+(nInt_J(nCl_J(V_1))) \cap F^-(nInt_J(nCl_J(V_2)))$. This implies that $F(U) \subset nInt_J(nCl_J(V_1))$ and $F(u) \cap nInt_J(nCl_J(V_2)) \neq \emptyset$ for each $u \in U$. Therefore, F is almost (mI, nJ) -continuous at $x \in X$. $\square$

Theorem 4.2. For a multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$, the following properties are equivalent:

- (1) F is almost (mI, nJ) -continuous;
 (2) for each $x \in X$ and all nJ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in mIO(X)$ containing x such that $F(U) \subset nInt_J(nCl_J(V_1))$ and $F(u) \cap nInt_J(nCl_J(V_2)) \neq \emptyset$ for every $u \in U$;
 (3) for each $x \in X$ and all nJ -regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$;
 (4) $F^+(G_1) \cap F^-(G_2) = mInt_I(F^+(G_1) \cap F^-(G_2))$ for all nJ -regular open sets G_1, G_2 of Y ;
 (5) $F^+(K_1) \cup F^-(K_2) = mCl_I(F^+(K_1) \cup F^-(K_2))$ for all nJ -regular closed sets K_1, K_2 of Y ;
 (6) $mCl_I(F^-(nCl_J(nInt_J(K_1))) \cup F^+(nCl_J(nInt_J(K_2)))) \subset F^-(K_1) \cup F^+(K_2)$ for all nJ -closed sets K_1 and K_2 of Y ;
 (7) $mCl_I(F^-(nCl_J(nInt_J(nCl_J(B_1)))) \cup F^+(nCl_J(nInt_J(nCl_J(B_2))))) \subset F^-(nCl_J(B_1)) \cup F^+(nCl_J(B_2))$ for all subsets B_1 and B_2 of Y .

Proof. (1) $\Rightarrow$ (2): The proof follows from Theorem 4.1.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (4): Let G_1, G_2 be any nJ -regular open sets of Y . Suppose that $x \in F^+(G_1) \cap F^-(G_2)$. Then $F(x) \in G_1^+ \cap G_2^-$ and there exists $U \in mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore, we have $x \in U \subset F^+(G_1) \cap F^-(G_2)$. Hence $F^+(G_1) \cap F^-(G_2) \subset mInt_I[F^+(G_1) \cap F^-(G_2)]$. We obtain $F^+(G_1) \cap F^-(G_2) = mInt_I[F^+(G_1) \cap F^-(G_2)]$.

(4) $\Rightarrow$ (1): Let V_1, V_2 be any nJ -open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, $nInt_J(nCl_J(V_1))$ and $nInt_J(nCl_J(V_2))$ are nJ -regular open sets of Y and

$$F(x) \in (nInt_J(nCl_J(V_1)))^+ \cap (nInt_J(nCl_J(V_2)))^-.$$

Then, by (4) we have

$$\begin{aligned} x \in F^+(nInt_J(nCl_J(V_1))) \cap F^-(nInt_J(nCl_J(V_2))) = \\ mInt_I[F^+(nInt_J(nCl_J(V_1))) \cap F^-(nInt_J(nCl_J(V_2)))]. \end{aligned}$$

There exists $U \in mIO(X)$ containing x such that $F(U) \subset nInt_J(nCl_J(V_1))$ and $F(u) \cap nInt_J(nCl_J(V_2)) \neq \emptyset$ for each $u \in U$. Therefore, F is almost (mI, nJ) -continuous.

(4) $\Leftrightarrow$ (5): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ and $F^-(Y - B) = X - F^+(B)$.

(5) $\Rightarrow$ (6): Let K_1 and K_2 be any nJ -closed sets. Then $nCl_J(nInt_J(K_1))$ and $nCl_J(nInt_J(K_2))$ are nJ -regular closed.

By (5) $mCl_I(F^-(nCl_J(nInt_J(K_1))) \cup F^+(nCl_J(nInt_J(K_2)))) = F^-(nCl_J(nInt_J(K_1))) \cup F^+(nCl_J(nInt_J(K_2))) \subset F^-(K_1) \cup F^+(K_2)$ for all nJ -closed sets K_1 and K_2 of Y ;

(6) $\Rightarrow$ (7): The proof is obvious.

(7) $\Rightarrow$ (5): Let K_1, K_2 be any nJ -regular closed sets in Y . Then

$$\begin{aligned} mCl_I(F^-(K_1) \cup F^+(K_2)) = \\ mCl_I(F^-(nCl_J(nInt_J(nCl_J(K_1)))) \cup F^+(nCl_J(nInt_J(nCl_J(K_2))))) \\ \subset F^-(nCl_J(K_1)) \cup F^+(nCl_J(K_2)) = F^-(K_1) \cup F^+(K_2). \end{aligned}$$

It follows that $mCl_I[F^-(K_1) \cup F^+(K_2)] = F^-(K_1) \cup F^+(K_2)$. $\square$

Corollary 4.3. *For a multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$, the following properties are equivalent:*

- (1) F is almost (mI, nJ) -continuous;
- (2) $F^+(G_1) \cap F^-(G_2)$ is mI -open set in X for all nJ -regular open sets G_1, G_2 of Y ;
- (3) $F^+(K_1) \cup F^-(K_2)$ is mI -closed in X for all nJ -regular closed sets K_1, K_2 of Y .

Proof. By Lemma 3.2, $mIO(X)$ has property $\mathcal{B}$ and the proof is obvious by (4) and (5) of Theorem 4.2. $\square$

Let $mIO(X) = \alpha IO(X) = \{\alpha I\text{-open sets of } X\}$ and $nJO(Y) = sJO(Y) = \{\text{semi-}J\text{-open sets of } Y\}$. Then by Theorem 4.1 we obtain the following corollary.

Corollary 4.4. *For a multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$, the following properties are equivalent:*

- (1) F is almost $(\alpha I, sJ)$ -continuous at $x \in X$;
- (2) for any sJ -regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists a $U \in \alpha IO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$;
- (3) $x \in \alpha \text{Int}_I(F^+(G_1) \cap F^-(G_2))$ for all sJ -regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$;
- (4) $x \in \alpha \text{Int}_I(F^+(\text{sInt}_J(\text{sCl}_J(V_1))) \cap F^-(\text{sInt}_J(\text{sCl}_J(V_2))))$ for all sJ -open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$.

Let $J = \{\emptyset\}$, then $B^* = \text{Cl}(B)$ and $\text{Cl}^* = \text{Cl}$. Then $(Y, \sigma, J) = (Y, \sigma)$. Then almost (mI, nJ) -continuous multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is almost mI -continuous multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma)$, that is,

Definition 4.2. [12] A multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be

- (1) almost mI -continuous at a point $x \in X$ if for all open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in mIO(X)$ containing x such that $F(U) \subset \text{Int}(\text{Cl}(V_1))$ and $F(u) \cap \text{Int}(\text{Cl}(V_2)) \neq \emptyset$ for every $u \in U$,
- (2) F is almost mI -continuous if it has the property at each point $x \in X$.

Corollary 4.5. [12] *For a multifunction $F : (X, \tau, I) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

- (1) F is almost mI -continuous;
- (2) for each $x \in X$ and all open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists $U \in mIO(X)$ containing x such that $F(U) \subset \text{Int}(\text{Cl}(V_1))$ and $F(u) \cap \text{Int}(\text{Cl}(V_2)) \neq \emptyset$ for every $u \in U$;
- (3) for each $x \in X$ and all regular open sets G_1, G_2 of Y such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in mIO(X)$ containing x such that $F(U) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$;
- (4) $F^+(G_1) \cap F^-(G_2) = m\text{Int}_I(F^+(G_1) \cap F^-(G_2))$ for all regular open sets G_1, G_2 of Y ;
- (5) $F^+(K_1) \cup F^-(K_2) = m\text{Cl}_I(F^+(K_1) \cup F^-(K_2))$ for all regular closed sets K_1, K_2 of Y ;
- (6) $m\text{Cl}_I(F^-(\text{Cl}(\text{Int}(K_1))) \cup F^+(\text{Cl}(\text{Int}(K_2)))) \subset F^-(K_1) \cup F^+(K_2)$ for all closed

sets K_1 and K_2 of Y ;

(7) $mCl_I(F^-(Cl(Int(Cl(B_1)))) \cup F^+(Cl(Int(Cl(B_2)))) \subset F^-(Cl(B_1)) \cup F^+(Cl(B_2))$
for all subsets B_1 and B_2 of Y .

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