

BOHR INEQUALITIES FOR A SUBCLASS OF CLOSE-TO-CONVEX HARMONIC MAPPINGS

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ABSTRACT. In this paper, we obtain several improved versions of the sharp Bohr radius for a subclass of close-to-convex harmonic mappings which satisfies a third-order differential inequality. We provide a discussion on the sharp Bohr radius for harmonic mappings from the viewpoint of certain proper combinations.

1. INTRODUCTION

Let $\mathcal{H}$ denote the class of complex valued harmonic functions f defined in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and normalized by $f(0) = f_z(0) - 1 = 0$. Also, let $\mathcal{H}^0 = \{f \in \mathcal{H} : f_{\bar{z}}(0) = 0\}$. Each function $f \in \mathcal{H}^0$ can be expressed as $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=2}^{\infty} b_n z^n \quad (1.1)$$

are analytic functions in $\mathbb{D}$. The Jacobian J_f of $f = h + \bar{g}$ is given by $J_f = |h'|^2 - |g'|^2$. By Lewy's theorem (see [8]), f is locally univalent and sense-preserving in $\mathbb{D}$, if and only if $J_f(z) > 0$, or equivalently, if $h' \neq 0$ and the dilation $\omega = g'/h'$ has the property $|\omega| < 1$ in $\mathbb{D}$. In the main results, we will investigate Bohr radius for subclasses of close-to-convex harmonic mappings. The related subclasses of harmonic mappings can be found in [16, 14, 13].

We denote by $\mathcal{B}$ the Bohr radius for analytic functions f in $\mathbb{D}$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}$. In 1914, Bohr [5] proved that if $f \in \mathcal{B}$ is of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the majorant series $M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n$ of f satisfies

$$M_{f_0}(r) = \sum_{n=1}^{\infty} |a_n| r^n \leq 1 - |a_0| \quad (1.2)$$

for all $z \in \mathbb{D}$ with $|z| = r \leq 1/6$, where $f_0(z) = f(z) - f(0)$. Later, Wiener, Riesz and Schur, independently improved the Bohr inequality (1.2) for $|z| \leq 1/3$ as the sharp Bohr radius for $f \in \mathcal{B}$. Formula (1.2) is called the Bohr inequality, $1/3$ is called the Bohr radius, and this phenomenon is called the Bohr phenomenon.

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There are several different types of Bohr radius. One kind of Bohr radius for analytic functions containing area elements was studied by Kayumov and Ponnusamy [10] as following.

Theorem 1.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$, $|f(z)| \leq 1$ and S_r denote the area of the image $|z| < r$ under the mapping f . Then*

$$B_1(r) := \sum_{n=0}^{\infty} |a_n| r^n + \frac{16}{9} \left(\frac{S_r}{\pi} \right) \leq 1 \text{ for } r \leq \frac{1}{3},$$

and the values $1/3$ and $16/9$ cannot be improved. Moreover,

$$B_2(r) := |a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \frac{9}{8} \left(\frac{S_r}{\pi} \right) \leq 1 \text{ for } r \leq \frac{1}{2},$$

and the values $1/2$ and $9/8$ cannot be improved.

Another type of Bohr radius called Bohr-Rogosinski radius, appeared in [10]. In fact, it is described as follows: if $f \in \mathcal{B}$, then for $N \geq 1$, we have $|S_N(z)| < 1$ in the disk $\mathbb{D}_{1/2}$ and this radius is sharp, where $S_N(z) = \sum_{n=0}^N a_n z^n$ denotes the partial sums of f . The Bohr-Rogosinski sum $R_N^f(z)$ of f defined by

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \quad (|z| = r). \quad (1.3)$$

Theorem 1.2. [10] *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| < 1$ in $\mathbb{D}$. Then*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq R_N,$$

where R_N is the positive root of the equation

$$2(1+r)r^N - (1-r^2) = 0.$$

The radius R_N is the best possible. Moreover,

$$|f(z)|^2 + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq R'_N,$$

where R'_N is the positive root of the equation

$$(1+r)r^N - (1-r^2) = 0.$$

The radius R'_N is the best possible.

Recently, Wang *et al.* defined a new subclass of close-to-convex harmonic mappings which satisfies a third-order differential inequality.

Definition 1.1. ([16, Theorem 3.4]) For $\alpha \geq \gamma \geq 0$ and $0 \leq \beta < 1$, let $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ denotes the class of harmonic mappings $f = h + \bar{g}$, which satisfy the

condition

$$\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma) = \left\{ f = h + \bar{g} \in \mathcal{H}^0 : \Re(h'(z) + \alpha z h''(z) + \gamma z^2 h'''(z) - \beta) \right. \\ \left. > |g'(z) + \alpha z g''(z) + \gamma z^2 g'''(z)| \quad (z \in \mathbb{D}) \right\}. \quad (1.4)$$

Theorem 1.3. ([16, Theorem 3.4]) *Let $f = h + \bar{g} \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$. Then for $n \geq 2$, we have*

$$|a_n| + |b_n| \leq \frac{2(1 - \beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}, \\ ||a_n| - |b_n|| \leq \frac{2(1 - \beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}, \\ |a_n| \leq \frac{2(1 - \beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}.$$

All these results are sharp for the function given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} z^n. \quad (1.5)$$

Theorem 1.4. ([16, Theorem 3.6]) *If $f = h + \bar{g} \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$, then*

$$|z| + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1 - \beta)|z|^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \leq |f(z)| \\ \leq |z| + \sum_{n=2}^{\infty} \frac{2(1 - \beta)|z|^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}. \quad (1.6)$$

Both the inequalities are sharp for the function given by (1.5) or its rotations.

Such results as coefficient bounds and growth estimates derived here can be seen as generalizations of recent works (see [6, 9, 11, 12, 15]).

In this paper, we mainly deal with the functions $f = h + \bar{g} \in \mathcal{H}^0$ of the class $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ which is defined by the third-order differential inequality (1.4). We establish various Bohr inequalities for the harmonic functions in a suitable fashion.

2. BOHR'S INEQUALITIES FOR THE SUBCLASS

Before stating the main results of this paper, for the sake of simplicity, we list the following notation concerning $f \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$, where $z \in \mathbb{D}$, $p > 0$ and $N \geq 2, 0 \leq t \leq 1$, which will be used in the main results.

$$\mathbb{I}_{\alpha, \beta, \gamma}^{p, N}(z) := |z| + |f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n \\ \mathbb{II}_{\alpha, \beta, \gamma}^{t, p, N, \mathcal{N}}(z) := |f(z)|^p + t \sqrt{J_f(z)} |z| + (1 - t) P\left(\frac{S_r}{\pi}\right) + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n.$$

The $\mathcal{N}^{\text{th}}$ degree polynomial in S_r/π of the form

$$P\left(\frac{S_r}{\pi}\right) = \left(\frac{S_r}{\pi}\right)^{\mathcal{N}} + \left(\frac{S_r}{\pi}\right)^{\mathcal{N}-1} + \cdots + \frac{S_r}{\pi},$$

where S_r denotes the image of the subdisk $|z| < r$ under mapping f .

The main aim of this section is to study the Bohr phenomenon for the class $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ for $\alpha \geq \gamma \geq 0$ and $0 \leq \beta < 1$. We find the improved Bohr radius as well as Bohr-Rogosinski inequality for functions in the class $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$.

Using Theorem 1.3 and Theorem 1.4, considering power of the coefficient $|a_n| + |b_n|$, we prove the following sharp Bohr radius for the class $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$.

Theorem 2.1. *Let $f \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$. Then for $N \geq 2$ and $p > 0$, we have*

$$\mathbb{I}_{\alpha, \beta, \gamma}^{p, N}(z) = |z| + |f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n \leq \text{dist}(f(0), \partial f(\mathbb{D})) \quad (2.1)$$

for $|z| = r \leq R_{p, N}$, where $R_{p, N}$ is the unique root of

$$\begin{aligned} F_1(r) := & r + \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p \\ & + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\ & - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} = 0. \end{aligned}$$

Here $R_{p, N}$ is the best possible.

Proof. Let $f \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ be given by (1.1). Then by Theorem 1.4 for $|z| = r$, we have

$$\begin{aligned} r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} & \leq |f(z)| \\ & \leq r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}. \end{aligned} \quad (2.2)$$

It is easy to see that $f(0) = 0$ and hence, $|f(z) - f(0)| = |f(z)|$. The Euclidean distance between $f(0)$ and the boundary of $f(\mathbb{D})$ is given by

$$d(f(0), \partial f(\mathbb{D})) = \liminf_{|z|=r \rightarrow 1^-} |f(z) - f(0)|. \quad (2.3)$$

It is easy to show by the Weierstrass Discriminance that the series

$$r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}$$

is absolutely and uniformly convergent, hence, the limit and summation sign can be interchanged. Therefore, in view of (2.3), we obtain

$$d(f(0), \partial f(\mathbb{D})) \geq 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}. \quad (2.4)$$

Using Theorems 1.3 and 1.4, for $|z| = r$, we obtain

$$\begin{aligned} I_{\alpha, \beta, \gamma}^{p, N}(z) &= |z| + |f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n \\ &\leq r + \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p \\ &\quad + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}. \end{aligned} \quad (2.5)$$

It is easy to see that

$$\begin{aligned} &r + \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p + \\ &\sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\ &\leq 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \end{aligned} \quad (2.6)$$

for $r \leq R_{p, N}$, where $R_{p, N}$ is the smallest root of

$$\begin{aligned} F_1(r) &:= r + \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p \\ &\quad + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\ &\quad - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} = 0 \end{aligned} \quad (2.7)$$

in $(0, 1)$.

It is easy to see that F_1 is strictly increasing in $[0, 1)$ for all $p > 0$. Moreover,

$$F_1(0) = -1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} < 0$$

and

$$\lim_{r \rightarrow 1} F_1(r) = +\infty.$$

This shows that $R_{p,N}$ is the unique solution of the equation $F_1(r) = 0$ in $(0, 1)$. Therefore, we must have

$$\begin{aligned}
& R_{p,N} + \left(R_{p,N} + \sum_{n=2}^{\infty} \frac{2(1-\beta)R_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p \\
& + \sum_{n=N}^{\infty} \frac{2(1-\beta)R_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\
& - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} = 0
\end{aligned} \tag{2.8}$$

which is equivalent to

$$\begin{aligned}
& R_{p,N} + \left(R_{p,N} + \sum_{n=2}^{\infty} \frac{2(1-\beta)R_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p \\
& + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\
& = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}.
\end{aligned} \tag{2.9}$$

In view of (2.4), (2.5) and (2.6) for $|z| = r \leq R_{p,N}$, it follows that

$$I_{\alpha,\beta,\gamma}^{f,p,N}(r) = |z| + |f(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(\mathbb{D})).$$

In order to show that the constant $R_{p,N}$ is best possible, we consider the function $f = f_{(\alpha,\beta,\gamma)}$ defined by

$$f_{(\alpha,\beta,\gamma)}(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\beta)z^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}. \tag{2.10}$$

It is easy to show that $f_{(\alpha,\beta,\gamma)} \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ and for $f = f_{(\alpha,\beta,\gamma)}$, a simple computation shows that

$$d(f_{(\alpha,\beta,\gamma)}(0), \partial f_{(\alpha,\beta,\gamma)}(\mathbb{D})) = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)z^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}. \tag{2.11}$$

For $f = f_{(\alpha, \beta, \gamma)}$ and $|z| = r > R_{p,N}$, using (2.9) and (2.11), a simple computation shows that

$$\begin{aligned}
I_{\alpha, \beta, \gamma}^{f, p, N}(r) &= |z| + |f_{(\alpha, \beta, \gamma)}(z)|^p + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n \\
&= r + \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \right)^p \\
&\quad + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \\
&> R_{p,N} + \left(R_{p,N} + \sum_{n=2}^{\infty} \frac{2(1-\beta)R_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \right)^p \\
&\quad + \sum_{n=N}^{\infty} \frac{2(1-\beta)R_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \\
&= 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \\
&= d(f_{(\alpha, \beta, \gamma)}(0), \partial f_{(\alpha, \beta, \gamma)}(\mathbb{D})).
\end{aligned} \tag{2.12}$$

Therefore, the radius $R_{p,N}$ is best possible. This completes the proof. $\square$

We prove the next result finding the sharp Bohr-Rogosinski radius.

Theorem 2.2. *Let $f \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$. Then for $N \geq 2$ and $p > 0, 0 \leq t \leq 1$, we have*

$$\begin{aligned}
\Pi_{\alpha, \beta, \gamma}^{t, p, N, \mathcal{N}}(z) &= |f(z)|^p + t\sqrt{J_f(z)}|z| + (1-t)P\left(\frac{S_r}{\pi}\right) \\
&\quad + \sum_{n=N}^{\infty} (|a_n| + |b_n|) |z|^n \leq \text{dist}(f(0), \partial f(\mathbb{D}))
\end{aligned} \tag{2.13}$$

for $|z| = r \leq \tilde{R}_{p,N}$, where $\tilde{R}_{p,N}$ is the unique root of

$$\begin{aligned}
&\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \right)^p \\
&\quad + t \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^2 + (\alpha - 3\gamma)n + (1-\alpha + 2\gamma)} \right) \\
&\quad + (1-t)P\left(r^2 + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n]^2} \right) \\
&\quad + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \\
&\quad - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} = 0.
\end{aligned} \tag{2.14}$$

Here $\tilde{R}_{p,N}$ is the best possible.

Proof. Let $f \in \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ be given by (1.1). It is well-known that (see [7], p. 113]) the area of the disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ under the harmonic mapping $f = h + \bar{g}$ is

$$S_r = \iint_{\mathbb{D}_r} J_f(z) dx dy = \iint_{\mathbb{D}_r} (|h'(z)|^2 - |g'(z)|^2) dx dy. \quad (2.15)$$

In view of Theorem 1.3 and (2.15), a simple computation shows that

$$\begin{aligned} \frac{S_r}{\pi} &= \frac{1}{\pi} \iint_{\mathbb{D}_r} (|h'(z)|^2 - |g'(z)|^2) dx dy \\ &= r^2 + \sum_{n=2}^{\infty} n (|a_n|^2 - |b_n|^2) r^{2n} \\ &\leq r^2 + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n]^2}. \end{aligned}$$

The Jacobian of functions

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 \leq |h'(z)|^2 \leq \left(1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^{n-1}}{\gamma n^2 + (\alpha - 3\gamma)n + (1 - \alpha + 2\gamma)}\right)^2.$$

In view of Theorems 1.3 and 1.4 for $|z| = r$, we obtain

$$\begin{aligned} \Pi_{\alpha, \beta, \gamma}^{t,p,N,\mathcal{N}}(z) &= |f(z)|^p + t\sqrt{J_f(z)}|z| + (1-t)P\left(\frac{S_r}{\pi}\right) + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n \\ &\leq \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}\right)^p \\ &\quad + t\left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^2 + (\alpha - 3\gamma)n + (1 - \alpha + 2\gamma)}\right) \\ &\quad + (1-t)P\left(r^2 + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n]^2}\right) \\ &\quad + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} =: \Theta(r). \end{aligned} \quad (2.16)$$

A simple computation shows that

$$\Theta(r) \leq 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}$$

for $r \leq \tilde{R}_{p,N}$, where $\tilde{R}_{p,N}$ is the smallest root of the equation

$$\begin{aligned}
& \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \right)^p \\
& + t \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^2 + (\alpha - 3\gamma)n + (1-\alpha + 2\gamma)} \right) \\
& + (1-t)P \left(r^2 + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n]^2} \right) \\
& + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \\
& - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)} = 0.
\end{aligned} \tag{2.17}$$

Let $F_2 : [0, 1) \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned}
& \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \right)^p \\
& + t \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^2 + (\alpha - 3\gamma)n + (1-\alpha + 2\gamma)} \right) \\
& + (1-t)P \left(r^2 + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n]^2} \right) \\
& + \sum_{n=N}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \\
& - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)}.
\end{aligned} \tag{2.18}$$

By the same argument applied in the case of F_1 , it is easy to show that $\tilde{R}_{p,N}$ is the unique root of $F_2(r) = 0$ in $(0, 1)$. Therefore, we have $F_2(\tilde{R}_{p,N}) = 0$ which is equivalent to

$$\begin{aligned}
& \left(\tilde{R}_{p,N} + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)\tilde{R}_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha + 2\gamma)n} \right)^p \\
& + t \left(\tilde{R}_{p,N} + \sum_{n=2}^{\infty} \frac{2(1-\beta)\tilde{R}_{p,N}^n}{\gamma n^2 + (\alpha - 3\gamma)n + (1-\alpha + 2\gamma)} \right)
\end{aligned}$$

$$\begin{aligned}
& + (1-t)P \left(\tilde{R}_{p,N}^{2n} + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 \tilde{R}_{p,N}^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n]^2} \right) \\
& + \sum_{n=N}^{\infty} \frac{2(1-\beta) \tilde{R}_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\
& = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}.
\end{aligned} \tag{2.19}$$

It is enough to show that the constant $\tilde{R}_{p,N}$ is best possible. Henceforth, we consider the function $f = f_{(\alpha,\beta,\gamma)}$ defined by (2.10). For $f_{(\alpha,\beta,\gamma)}$ and $z = r > \tilde{R}_{p,N}$, a simple computation using (2.11) and (2.19) shows that

$$\begin{aligned}
\Pi_{\alpha,\beta,\gamma}^{f,p,N}(r) & = |f_{(\alpha,\beta,\gamma)}|^p + t\sqrt{J_{f_{(\alpha,\beta,\gamma)}}(z)}|z| + (1-t)P \left(\frac{S_r}{\pi} \right) + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n \\
& = \left(r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p \\
& \quad + t \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^2 + (\alpha - 3\gamma)n + (1 - \alpha + 2\gamma)} \right) \\
& \quad + (1-t)P \left(r^2 + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n]^2} \right) \\
& \quad + \sum_{n=N}^{\infty} \frac{2(-1)^{n-1}(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\
& > \left(\tilde{R}_{p,N} + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)\tilde{R}_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \right)^p \\
& \quad + t \left(1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)\tilde{R}_{p,N}^{2n}}{\gamma n^2 + (\alpha - 3\gamma)n + (1 - \alpha + 2\gamma)} \right) \\
& \quad + (1-t)P \left(\tilde{R}_{p,N}^2 + \sum_{n=2}^{\infty} \frac{4n(1-\beta)^2 \tilde{R}_{p,N}^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n]^2} \right) \\
& \quad + \sum_{n=N}^{\infty} \frac{2(-1)^{n-1}(1-\beta)\tilde{R}_{p,N}^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\
& = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} \\
& = d(f_{(\alpha,\beta,\gamma)}(0), \partial f_{(\alpha,\beta,\gamma)}(\mathbb{D})).
\end{aligned} \tag{2.20}$$

This completes the proof. $\square$

Remark 2.1. Many previous results can be obtained from Theorems 2.1 and 2.2 by taking different variable values of $\alpha, \beta, \gamma, t, p, N, \mathcal{N}$. For example,

- When $\alpha = \gamma = 0$, the result of Theorem 2.1 yields the corresponding result of [1, Theorem 9].
- When $\alpha = \gamma = 0$, for $t = 0, \mathcal{N} = 2$ and $t = 1$, the results of Theorem 2.2 yields the corresponding results of the class $\mathcal{W}_{\mathcal{H}}^0(0, \beta, 0)$ in [1, Theorems 10 and 11], respectively.
- When $\alpha = \gamma = 0$, for $p = 1, t = 1$, the results of Theorem 2.2 yields the corresponding results of [4, Theorem 5.1].
- When $\gamma = 0$, for $p = 2, t = 0, N = 2, \mathcal{N} = 3$, the results Theorem 2.2 yields the corresponding results of the class $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, 0)$ in [3, Theorem 2.3(ii)].

Remark 2.2. Various special subclasses can be obtained from the class $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ by taking different variable values of α, β, γ . Such as, when $\gamma = 0$, the class $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$ yields the corresponding subclasses of $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, 0) = \mathcal{W}_{\mathcal{H}}^0(\alpha, \beta)$ which is considered by [2, 14].

Let $\alpha = 2, \gamma = \frac{1}{2}, \beta = \frac{1}{2}, p = 2, N = 2, t = \frac{1}{2}, \mathcal{N} = 2$ in Theorems 2.1 and 2.2, respectively, we get the following corollaries.

Corollary 2.3. Let $f \in \mathcal{W}_{\mathcal{H}}^0(2, \frac{1}{2}, \frac{1}{2})$. Taking $p = 2, N = 2$. Then

$$|z| + |f(z)|^2 + \sum_{n=2}^{\infty} (|a_n| + |b_n|) |z|^n \leq \text{dist}(f(0), \partial f(\mathbb{D})) \quad (2.21)$$

for $|z| \leq R_1 \approx 0.507020$, where R_1 is the unique root of

$$r + \left(r + \sum_{n=2}^{\infty} \frac{2r^n}{n^3 + n^2} \right)^2 + \sum_{n=2}^{\infty} \frac{2r^n}{n^3 + n^2} - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{n^3 + n^2} = 0.$$

Here R_1 is the best possible.

Corollary 2.4. Let $f \in \mathcal{W}_{\mathcal{H}}^0(2, \frac{1}{2}, \frac{1}{2})$. Taking $p = 2, N = 2, t = \frac{1}{2}, \mathcal{N} = 2$. Then

$$|f(z)|^2 + \frac{1}{2} \sqrt{J_f(z)} |z| + \frac{1}{2} \left(\left(\frac{S_r}{\pi} \right)^2 + \frac{S_r}{\pi} \right) + \sum_{n=2}^{\infty} (|a_n| + |b_n|) |z|^n \leq \text{dist}(f(0), \partial f(\mathbb{D})) \quad (2.22)$$

for $|z| \leq R_2 \approx 0.534062$, where R_2 is the unique root of

$$\begin{aligned} & \left(r + \sum_{n=2}^{\infty} \frac{2r^n}{n^2(n+1)} \right)^2 + \frac{1}{2} \left(r + \sum_{n=2}^{\infty} \frac{2r^n}{n(n+1)} \right) + \frac{1}{2} \left[\left(r^2 + \sum_{n=2}^{\infty} \frac{4r^{2n}}{n^3(n+1)^2} \right)^2 \right. \\ & \left. + \left(r^2 + \sum_{n=2}^{\infty} \frac{4r^{2n}}{n^3(n+1)^2} \right) \right] + \sum_{n=2}^{\infty} \frac{2r^n}{n^2(n+1)} - 1 - \sum_{n=2}^{\infty} (-1)^{n-1} \frac{2}{n^2(n+1)} = 0. \end{aligned}$$

Here R_2 is the best possible.

CONCLUSION

In this paper, we established the Bohr-type inequalities for the subclass of close-to-convex harmonic mappings $\mathcal{W}_{\mathcal{H}}^0(\alpha, \beta, \gamma)$, defined by a third-order differential inequality. Specifically, we proved sharp Bohr radius results and sharp Bohr–Rogosinski radius results for functions in this class. The radii are shown to be best possible, as demonstrated by extremal functions. Our results unify and extend several recent findings in the literature, corresponding to various choices of the parameters $\alpha, \beta, \gamma, t, p, N$ and $\mathcal{N}$. As natural directions for future research, one may explore similar Bohr phenomena for other subclasses of harmonic mappings defined by higher-order differential operators, or investigate refined inequalities involving area and Jacobian terms in more general settings.

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