

AN EXTENSION OF JANOWSKI FUNCTIONS WITH APPLICATIONS

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IN HONOUR OF PROFESSOR T. O. OPOOLA.

ABSTRACT. Janowski functions are regular functions of the form $p(z) = 1 + p_1z + \dots$, which are subordinate to $L_0(a, b : z) = (1 + az)/(1 + bz)$ for real numbers a, b such that $-1 \leq b < a \leq 1$. In this paper, we expand the space of the constants, a and b , of the superordinate function, $L_0(a, b : z)$, to a larger set of complex numbers and provide some applications.

1. INTRODUCTION

The greatly generalized family of Janowski functions consists of functions $p(z) = 1 + p_1z + \dots$, which are regular in the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ and are subordinate to $L_0(a, b : z) = (1 + az)/(1 + bz)$ for real numbers a, b such that $-1 \leq b < a \leq 1$ [12]. The important roles of the classical Janowski functions in the theory of regular schlicht functions are well known. Practically most classes of geometric functions have representation in terms of them, with the various choices of the parameters, a and b , of the superordinate (dominant) function,

$$L_0(a, b : z) = \frac{1 + az}{1 + bz}, \quad -1 \leq b < a \leq 1.$$

See [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 16] for example for many works associated with the Janowski functions.

It is however well known that the parameters a and b of the Janowski maps have been restricted to the set of real numbers only as defined above. The object of this paper, therefore, is to extend the space of the parameters a and b to a well-defined larger set of complex numbers. The extension of these parameters to the set of complex numbers has provided more possibilities and insights into the study of functions which are analytic and univalent in the unit disk. New relationships between the extended Janowski functions and other classes of functions yielding many new classes of applicable analytic and univalent functions have been identified.

It is also important to remark that these new relationships shall open a floodgate to many new researchable problems in the theory of regular schlicht functions. Much

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future research will spring up from the new extension of the parameters. We shall demonstrate this with some new definitions of classes of functions.

We begin by carefully defining a non-vacuous concept as follows:

Let s and t be complex numbers such that:

- (i) $|s| \leq 1$, $|t| \leq 1$ and
- (ii) $\operatorname{Re} t < \operatorname{Re} s$.

We affirm that the concept is non-vacuous as it is not difficult to see from the following example:

Example 1.1. For real numbers a, b, λ such that $-1 \leq b < a \leq 1$ and $\lambda \in (-1, 1)$, define

$$s = \frac{a + i\lambda b}{1 + i\lambda}, \quad t = \frac{b + i\lambda a}{1 + i\lambda} \quad (1.1).$$

Proof. By simple computations, given the conditions on the parameters a, b and λ , we can see easily that s and t satisfy $|s| \leq 1$, $|t| \leq 1$. Similarly we find that

$$\operatorname{Re} s = \frac{a + \lambda^2 b}{1 + \lambda^2}, \quad \operatorname{Re} t = \frac{b + \lambda^2 a}{1 + \lambda^2}$$

so that

$$\operatorname{Re} s - \operatorname{Re} t = \frac{(a - b)(1 - \lambda^2)}{1 + \lambda^2} > 0$$

which shows that $\operatorname{Re} t < \operatorname{Re} s$ as desired. $\square$

The couple (1.1) is a generating string for the complex numbers s and t .

Now let Ω be the class of functions $\omega(z) = b_1 z + b_2 z^2 + \dots$ which are also regular E and satisfy $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in E$. Then for complex numbers s and t as defined in (1.1), we say:

DEFINITION. $P[s, t]$ is the family of functions, $p(z)$, normalized by $p(0) = 1$, regular in E and such that for some $\omega \in \Omega$,

$$p(z) = \frac{1 + s\omega(z)}{1 + t\omega(z)}, \quad z \in E. \quad (1.2)$$

Equivalently $P[s, t]$ is the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

such that

$$p(z) \prec L_0(s, t : z) = \frac{1 + sz}{1 + tz}, \quad z \in E$$

where $\prec$ denotes subordination.

Theorem 1.2. With the complex numbers s and t as defined in (1.1),

$$L_0(s, t : z) = \frac{1 + sz}{1 + tz}$$

is well defined (normalized, has positive real part and maps the unit disk E onto a convex domain).

Proof. For some real numbers a, b such that $-1 \leq b < a \leq 1$, it is well known that $L_0(a, b : z) = (1 + az)/(1 + bz)$ is normalized, has positive real part and maps the unit disk E onto a convex domain. Then it is sufficient to show that $L_0(s(\lambda), t(\lambda) : z)$ is a linear transformation of the half-plane defined by $L_0(a, b : z)$ onto itself [10]. Since $s(\lambda)$ and $t(\lambda)$ satisfy the required conditions and so are well defined, then we have

$$\begin{aligned} L_0(s, t : z) &= \frac{\frac{1+az}{1+bz} + i\lambda}{1 + i\lambda \frac{1+az}{1+bz}} \\ &= \frac{L_0(a, b : z) + i\lambda}{1 + i\lambda L_0(a, b : z)}. \end{aligned}$$

which is the linear transformation of the half-plane defined by $L_0(a, b : z)$ onto itself as required. $\square$

Since $L_0(s, t : z)$ is a linear transformation of the half-plane defined by $L_0(a, b : z)$ onto itself, an important corollary to the above result is:

Corollary 1.3. *Functions in $P[s, t]$ have positive real part.*

The classes $P[s, t]$ correspond to the well-known Janowski classes of functions, $P[a, b]$, if $s = a$ and $t = b$, and this happens when $\lambda = 0$. In particular, if $s = a = 1$ and $t = b = -1$, we have the class of Carathéodory functions P consisting of functions $h(z) = 1 + c_1z + c_2z^2 + \dots$. Many special cases of $P[s, t]$ can also be derived as would be shown later.

Since $h = (1 + \omega(z))/(1 + \omega(z))$, then from (1.2), easy but careful computation shows that:

Theorem 1.4. *For some $p \in P[s, t]$ and $h \in P$, the following relation holds:*

$$p(z) = \frac{(1 + s)h(z) + 1 - s}{(1 + t)h(z) + 1 - t}.$$

Theorem 1.5. *If $p \in P[s, t]$ then $|p_k| \leq |s - t|$ for $k = 1, 2, 3, \dots$. Equality holds for $L_0(s, t : z) = (1 + sz)/(1 + tz)$.*

Proof. Following Goel and Mehrotra [8], then from (1.2), we have

$$1 + s\omega(z) = (1 + t\omega(z))p(z) = (1 + t\omega(z)) \left(1 + \sum_{k=1}^{\infty} p_k z^k \right)$$

so that

$$\sum_{k=1}^{\infty} p_k z^k = \omega(z) \left((s - t) - t \sum_{k=1}^{\infty} p_k z^k \right).$$

By a lemma of Robertson (see [8], Lemma 2), we have

$$\sum_{k=1}^n |p_k|^2 \leq |s - t|^2 + |t| \sum_{k=1}^{n-1} |p_k|^2.$$

That is,

$$|p_n|^2 \leq |s - t|^2 - (1 - |t|) \sum_{k=1}^{n-1} |p_k|^2 \leq |s - t|^2$$

which proves the inequality. □

Theorem 1.6. *If $p \in P[s, t]$ then*

$$|p(z)| \leq \frac{1 + |s|r}{1 - |t|r}.$$

Proof. From (1.2), we have

$$p(z) = \frac{1 + s\omega(z)}{1 + t\omega(z)}.$$

Expanding in series of $\omega(z)$, we have

$$p(z) = 1 + (s - t)\omega(z) + t(s - t)\omega(z)^2 + t^2(s - t)\omega(z)^3 + \cdots.$$

That is,

$$p(z) = 1 + (s - t) \sum_{k=1}^{\infty} t^{k-1} \omega(z)^k$$

so that

$$|p(z)| \leq 1 + |s - t| \sum_{k=1}^{\infty} |t|^{k-1} |\omega(z)|^k.$$

Since $|\omega(z)| \leq |z| = r$, we have

$$|p(z)| \leq 1 + |s - t| \sum_{k=1}^{\infty} |t|^{k-1} r^k = 1 + |s - t| \frac{r}{1 - |t|r} \leq \frac{1 + |s|r}{1 - |t|r}$$

as required. □

Theorem 1.7. *If $p \in P[s, t]$. Define*

$$q(z) = \int_0^z p(\rho) d\rho.$$

Then

$$|q(z)| \leq -\frac{|s|}{|t|} r - \frac{|s| + |t|}{|t|^2} \ln(1 - |t|r).$$

Proof. Given that

$$q(z) = \int_0^z p(\rho) d\rho,$$

then letting $\rho = \sigma e^{i\theta}$. Then we have

$$|q(z)| \leq \int_0^\rho |p(\sigma e^{i\theta})| d\sigma$$

Thus we have

$$|q(z)| \leq \int_0^{|z|} \frac{1 + |s|\sigma}{1 - |t|\sigma} d\sigma$$

which by integration proves the theorem. □

We give examples of special cases of classes of functions packed in $P[s, t]$ for specific choices of s and t defined by (1.1).

The following are a few:

- (i) $P[1, -1] \equiv P$, the class of Caratheodory functions for which $\operatorname{Re} p(z) > 0$,
- (ii) $P[1 - 2\beta, -1] \equiv P_\beta$, the class of Caratheodory functions of order β , $0 \leq \beta < 1$, that is $\operatorname{Re} p(z) > \beta$,
- (iii) $P[1, 1/\beta - 1] \equiv P(\beta)$, the class of functions $p(z)$ satisfying $|p(z) - \beta| < \beta$, $\beta > 1/2$,
- (iv) $P[\beta, -\beta] \equiv P^{(\beta)}$, the class of functions $p(z)$ satisfying $|p(z) - 1| < \beta|p(z) + 1|$, $0 < \beta \leq 1$
- (v) $P[\beta, 0] \equiv P_{(\beta)}$, the class of functions $p(z)$ satisfying $|p(z) - 1| < \beta$, $0 < \beta \leq 1$.

Furthermore for real numbers $\lambda \in (-1, 1)$ and β , we have the following examples:

- (vi) $P\left[\frac{1-i\lambda}{1+i\lambda}, \frac{i\lambda-1}{1+i\lambda}\right]$ consisting of functions $p(z) = 1 + p_1z + \dots$ such that

$$p(z) \prec \frac{1 + i\lambda + (1 - i\lambda)z}{1 + i\lambda + (i\lambda - 1)z}. \quad (1.3)$$

- (vii) $P\left[\frac{1-2\beta-i\lambda}{1+i\lambda}, \frac{i\lambda(1-2\beta)-1}{1+i\lambda}\right]$ consisting of functions $p(z) = 1 + p_1z + \dots$ such that

$$p(z) \prec \frac{1 + i\lambda + (1 - 2\beta - i\lambda)z}{1 + i\lambda + (i\lambda(1 - 2\beta) - 1)z}, \quad 0 \leq \beta < 1. \quad (1.4)$$

- (viii) $P\left[\frac{\beta+i\lambda(1-\beta)}{\beta(1+i\lambda)}, \frac{1-\beta+i\beta\lambda}{\beta(1+i\lambda)}\right]$, $\beta > 1/2$, consisting of functions $p(z) = 1 + p_1z + \dots$ such that

$$p(z) \prec \frac{\beta(1 + i\lambda) + [\beta + i\lambda(1 - \beta)]z}{\beta(1 + i\lambda) + [(1 - \beta) + i\beta\lambda]z}, \quad \beta > 1/2. \quad (1.5)$$

- (ix) $P\left[\frac{\beta(1-i\lambda)}{1+i\lambda}, \frac{\beta(i\lambda-1)}{1+i\lambda}\right]$ consisting of functions $p(z) = 1 + p_1z + \dots$ such that

$$p(z) \prec \frac{1 + i\lambda + \beta(1 - i\lambda)z}{1 + i\lambda + \beta(i\lambda - 1)z}, \quad 0 < \beta \leq 1. \quad (1.6)$$

- (x) $P\left[\frac{\beta}{1+i\lambda}, \frac{i\beta\lambda}{1+i\lambda}\right]$ consisting of functions $p(z) = 1 + p_1z + \dots$ such that

$$p(z) \prec \frac{1 + i\lambda + \beta z}{1 + i\lambda + i\beta\lambda z}, \quad 0 < \beta \leq 1. \quad (1.7)$$

2. NEW SUBCLASSES OF FUNCTIONS ASSOCIATED WITH $P[s, t]$

Let A be the class of functions $f(z) = z + a_2z^2 + \dots$ which are analytic in the unit disk E . Denote by $\phi(z)$ the superordinate functions in Equations (1.3) to (1.7), and define

$$\begin{aligned} f'(z) &\prec \phi(z), \\ \frac{zf'(z)}{f(z)} &\prec \phi(z) \end{aligned}$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),$$

for functions $f \in A$. Then these classes of functions are respectively new subclasses of functions of bounded turning, starlike and convex functions, which are associated with the extended family $P[s, t]$ of functions having complex parameters.

The expositions of new subclasses of starlike and convex functions (and many more function classes) associated with the extended family $P[s, t]$ will be of independent interest. However, in this section, we shall begin the discussion of the first of these new subclasses, that is, new subclasses of functions of bounded turning associated with the extended family $P[s, t]$ of functions having complex parameters, as follows:

DEFINITION. A function $f \in A$ is said to belong to the class $B(\beta, \lambda)$ if and only if

$$f'(z) \prec \frac{1 + i\lambda + \beta z}{1 + i\lambda + i\beta\lambda z}, \quad 0 < \beta \leq 1, \quad z \in E.$$

The class $B(\beta, \lambda)$ is well-defined. The following function:

$$f(z) = -\frac{i}{\lambda}z + \frac{\lambda^2 - i}{\beta\lambda^2} \ln \left(1 + \frac{\beta\lambda}{1 + i\lambda}z \right) = z + \frac{1}{2} \frac{\beta(i\lambda - 1)}{1 + i\lambda} z^2 + \dots \quad (2.1)$$

whose derivative is given by

$$\phi(z) = \frac{1 + i\lambda + \beta z}{1 + i\lambda + i\beta\lambda z}$$

is in $B(\beta, \lambda)$.

The class $B(\beta, \lambda)$ is a subclass of the class, R , of functions whose derivatives have positive real parts in the unit disk E [15] as we shall prove next:

Theorem 2.1. $B(\beta, \lambda) \subset R$, that is, $B(\beta, \lambda)$ consists of univalent functions only.

Proof. It is sufficient to prove that the superordinate function

$$\phi(z) = \frac{1 + i\lambda + \beta z}{1 + i\lambda + i\beta\lambda z}, \quad 0 < \beta \leq 1, \quad z \in E$$

has positive real parts in E , that is, $\operatorname{Re} \phi(z) > 0$. Let $\alpha(z) = 1 + \beta z$. Then we have

$$2\operatorname{Re} \phi(z) = \frac{i\lambda + \alpha}{1 + i\lambda\alpha} + \frac{\bar{\alpha} - i\lambda}{1 - i\lambda\bar{\alpha}}.$$

Simplifying, we have

$$2\operatorname{Re} \phi(z) = \frac{2(1 + \lambda^2)\operatorname{Re} \alpha}{|1 + i\lambda\alpha|^2}.$$

Since $z \in E$, it is easy to see that $\operatorname{Re} \alpha(z) = \operatorname{Re}(1 + \beta z) > 0$, and so $\operatorname{Re} \phi(z) > 0$. $\square$

Theorem 2.2. If $f \in B(\beta, \lambda)$, then $|a_k| \leq \beta/k$. Equality is attained by the function $f(z)$ given by (2.1).

Proof. Since $f \in B(\beta, \lambda)$, then there exists $p \in P[s, t]$ with $s = \frac{\beta}{1+i\lambda}$ and $t = \frac{i\beta\lambda}{1+i\lambda}$ such that

$$f'(z) = p(z).$$

Hence, we have

$$|ka_k| \leq |s - t| = \beta$$

so that $|a_k| \leq \beta/k$. □

Theorem 2.3. *If $f \in B(\beta, \lambda)$ then*

$$|f'(z)| \leq \frac{1 + \lambda^2 + \beta r}{1 + \lambda^2 - \beta\lambda r}.$$

Equality is also attained by the function $f(z)$ given by (2.1).

Proof. Since $f \in B(\beta, \lambda)$, then there exists $p \in P[s, t]$ with $s = \frac{\beta}{1+i\lambda}$ and $t = \frac{i\beta\lambda}{1+i\lambda}$ such that

$$f'(z) = p(z).$$

Hence, we have

$$|f'(z)| = |p(z)| \leq \frac{1 + |s|r}{1 - |t|r}$$

so that using $|s| = \beta/(1 + \lambda^2)$ and $|t| = \beta\lambda/(1 + \lambda^2)$, we have the inequality as desired. □

Theorem 2.4. *Let $f \in B(\beta, \lambda)$. Then*

$$\begin{aligned} |f(z)| &\leq -\frac{1}{\lambda}r - \frac{(1 + \lambda)(1 + \lambda^2)}{\beta\lambda^2} \ln \left(1 - \frac{\beta\lambda}{1 + \lambda^2}r \right) \\ &= r + \frac{\beta(1 + \lambda)}{2(1 + \lambda^2)}r^2 + \dots \end{aligned}$$

Equality is also attained by the function $f(z)$ given by (2.1).

Proof. This follows from Theorem 1.7 taking $f(z) = q(z) = \int_0^z p(\rho)d\rho$ and using $|s| = \beta/(1 + \lambda^2)$ and $|t| = \beta\lambda/(1 + \lambda^2)$ as required. □

3. THE CASE $\lambda = 0$

The class $B(\beta, 0)$ is the subclass of functions $f(z)$ such that

$$|f'(z) - 1| < \beta, \quad 0 < \beta \leq 1, \quad z \in E.$$

The following corollaries follow by taking $\lambda = 0$ in $B(\beta, \lambda)$:

Corollary 3.1. *If $f \in B(\beta, 0)$, then $|a_k| \leq \beta/k$. Equality is attained by the function $f(z)$ given by:*

$$f(z) = z + \frac{\beta}{2}z^2 + \dots \quad (2.2)$$

Corollary 3.2. *If $f \in B(\beta, 0)$ then*

$$|f'(z)| \leq 1 + \beta r.$$

Equality is also attained by the function $f(z)$ given by (2.2).

Corollary 3.3. *Let $f \in B(\beta, 0)$. Then*

$$|f(z)| \leq r + \frac{\beta}{2}r^2 + \cdots.$$

Equality is also attained by the function $f(z)$ given by (2.2).

4. CONCLUDING REMARKS

The study of many subclasses of functions opened up in this paper such as the subclasses of starlike and convex functions associated with Janowski functions $P[s, t]$ having complex parameters shall make fruitful engagements for interested researchers in the near and distant future.

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