

FRACTIONAL CALCULUS WITHIN TWO BANACH SPACES AND AMPLIFIED NEURAL NETWORK APPROXIMATION

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ABSTRACT. Here we deal with the univariate and abstract multi-composite specific sigmoid activated quantitative approximation of Banach space valued continuous functions on a compact interval or on a Banach space. These happen by quasi-interpolation Banach space valued specific multi-composite sigmoid activated neural network operators. We perform also the related Banach space valued and Banach space originating specific multi-composite sigmoid activated fractional and abstract approximations. These multi-composite sigmoid activated infinitely many approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative or fractional derivatives. Our operators are defined by using a multi-composite density function induced by seven known and basic sigmoid functions. The approximations are pointwise and with respect to the uniform norm. The related Banach space valued and Banach space originating feed-forward neural networks are with one hidden layer.

1. INTRODUCTION

Multi-composition of activation functions in neural networks involves applying different activation functions across layers or even within the same layer to introduce varied non-linearities, allowing models to learn complex patterns better, often seen in advanced architectures like: Long Short-Term Memory (LSTM) networks or through techniques like "Activation Ensembles" to boost performance by blending functions (e.g., Sigmoid, Tanh, ReLU) for improved learning and tackling issues like vanishing gradients, leading to stronger, more versatile models than a single function might provide. The benefits follow:

Enhanced Non-linearity: Different functions capture different types of non-linear relationships, allowing the network to model more intricate data.

Improved Performance: Ensembles of different activation functions often outperform networks using just one, especially on complex tasks or smaller datasets.

Specialized Functions: Combining standard functions (like ReLU for speed) with others (like Sigmoid/Tanh for range control) offers flexibility.

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Applications:

Recurrent Networks: LSTM networks inherently use multiple functions (sigmoid, tanh) to manage cell state and gates.

Advanced Ensembles: Methods like "Activation Ensembles" learn weights for multiple functions at each neuron, creating a richer activation.

Hybrid Layers: Concatenating outputs from different activation functions within a layer or using them across different parts of a deep network.

As a conclusion: While a single, well-chosen function (like ReLU) often suffices, using multiple activations, either sequentially or combined, provides a powerful way to boost a neural network's representational power and accuracy, moving beyond simpler "vanilla" network designs.

In this article we are greatly inspired by the related works [5]-[7].

In Section 2 the author performs multicomposition of specific sigmoid functions and neural network univariate approximations to continuous functions over compact intervals of the real line or over a Banach space with values to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X -valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X -valued high order derivative, or X -valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. In preparation to present our results we establish important properties of the multicomposite specific density functions defining our infinitely many specific operators which are induced by multi-composition of seven basic and known sigmoid functions.

Feed-forward X -valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x , and σ is the multicomposition activation function of the network.

In Section 3 the author introduces Fractional Calculus for functions with domain a Banach space and values in a Banach space, for the first time in the literature, all based on [13] - [15]. He presents all the basics of a fractional calculus theory including left and right fractional Taylor's formulae.

In section 4 the author demonstrates his main results, that is applying all the results of Section 2 under the spirit of Section 3. Essential for this work are [1] - [4]. For basics about neural networks we recommend [8], [9], [11].

2. MULTICOMPOSITE NEURAL NETWORKS APPROXIMATION

We need

Definition 2.1. ([4], p. 530) The general sigmoid activation functions we deal with here are described as follows $h : \mathbb{R} \rightarrow [-1, 1]$, strictly increasing $h(0) = 0$, $h(-x) = -h(x)$, $x \in \mathbb{R}$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

More precisely we will deal with the following sigmoid activation functions which possess all of the above properties: the algebraic activation function,

$$g_1(x) = \frac{x}{\sqrt[2m]{1+x^{2m}}}, \quad m \in \mathbb{N}, x \in \mathbb{R}, \quad (1)$$

(see [3], pp. 2-3);

the normalized and parametrized arctangent

$$g_2(x) = \frac{2}{\pi} \arctan\left(\frac{x}{2}\lambda x\right) = \frac{2}{\pi} \int_0^{\frac{\pi\lambda x}{2}} \frac{dz}{1+z^2}, \quad x \in \mathbb{R}, \lambda > 0, \quad (2)$$

(see [4], p.p. 156-157);

the normalized and parametrized Gudermannian function

$$\begin{aligned} g_3(x) &= \frac{2}{\pi} gd(\lambda x) = \frac{4}{\pi} \arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right) \\ &= \frac{2}{\pi} \int_0^{\lambda x} \frac{dt}{\cosh t} = \frac{4}{\pi} \int_0^{\lambda x} \frac{dt}{e^t + e^{-t}}, \quad x \in \mathbb{R}, \lambda > 0, \end{aligned} \quad (3)$$

where

$$gd(x) := \int_0^x \frac{dt}{\cosh t} = 2 \arctan\left(\tanh\left(\frac{x}{2}\right)\right), \quad \forall x \in \mathbb{R},$$

is the Gudermannian function (see [4], p. 198-199);

the parametrized error function

$$g_4(x) = erf \lambda x = \frac{2}{\sqrt{\pi}} \int_0^{\lambda x} e^{-t^2} dt, \quad x \in \mathbb{R}, \lambda > 0, \quad (4)$$

(see [4], pp. 226-227);

the parametrized hyperbolic tangent function

$$\begin{aligned} g_5(x) &= \tanh \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}} = \frac{e^{2\lambda x} - 1}{e^{2\lambda x} + 1} \\ &= \frac{1 - e^{-2\lambda x}}{1 + e^{-2\lambda x}}, \quad x > 0, \lambda > 0, \end{aligned} \quad (5)$$

(see [4], p. 72);

for small $0 < \lambda < 1$, $\tanh \lambda x$ is expected to behave better than *ReLU* and *Leaky ReLU* activation functions;

the parametrized hyperbolic tangent like activation function,

$$g_6(x) = \frac{A^{\lambda x} - A^{-\lambda x}}{A^{\lambda x} + A^{-\lambda x}}, \quad A > 1, \lambda > 0, x \in \mathbb{R} \quad (6)$$

(see [4], pp. 262-263);

finally, the β -parametrized half hyperbolic tangent function

$$g_7(x) = \frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}, \quad \beta > 0, \forall x \in \mathbb{R}, \quad (7)$$

(see [4], p. 509-510).

So, we denote them by $g_i(x)$, $x \in \mathbb{R}$, $i = 1, 2, \dots, 7$.

Here we deal with the specific sequences of activation functions $(h_i^{(q)})$, $q \in \mathbb{N}$, $i \in \mathbb{N}$, such that all $h_i^{(q)} \in G := \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$. That is $h_i^{(q)}$ for different $i \in \mathbb{N}$ could be identical, $q \in \mathbb{N}$.

Therefore in the sequences of activation functions $h_1^{(q)}, h_2^{(q)}, \dots$, we allow random repetitions. Thus any two random $h_i^{(q)}$ activation functions are not necessarily different, but all belong to G . Call $H^{(q)} = \{h_1^{(q)}, h_2^{(q)}, \dots\}$, $q \in \mathbb{N}$, and by $\Omega := \{H^{(q)}, \text{ all } q \in \mathbb{N}\}$. For simplicity next we can consider such a sequence $\{h_1, h_2, \dots\}$.

Notice here $0 < h_i(1) \leq 1$, $i = 1, 2, \dots$. Any composition $h_1 \circ h_2 \circ h_3 \circ \dots \circ h_\lambda$ is meant to be $h_1|_{[-1,1]} \circ h_2|_{[-1,1]} \circ h_3|_{[-1,1]} \circ \dots \circ h_{\lambda-1}|_{[-1,1]} \circ h_\lambda$, $\lambda \in \mathbb{N}$, and for convenience, we denote it by $G_\lambda := h_1 \circ h_2 \circ h_3 \circ \dots \circ h_\lambda$. We have for any $\lambda \in \mathbb{N} : 0 < h_\lambda(1) \leq 1$, hence $0 < h_{\lambda-1}(h_\lambda(1)) \leq h_{\lambda-1}(1) \leq 1$, and $0 < h_{\lambda-2}(h_{\lambda-1}(h_\lambda(1))) \leq h_{\lambda-2}(h_{\lambda-1}(1)) \leq h_{\lambda-2}(1) \leq 1$.

Inductively we derive that $0 < G_\lambda(1) \leq 1$, $\forall \lambda \in \mathbb{N}$.

Clearly, it gives us $G_\lambda(0) = 0$ and G_λ is strictly increasing over $\mathbb{R}$. Furthermore it holds

$$\begin{aligned} G_\lambda(-x) &= h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(-x))))) = h_1(h_2(h_3(\dots(h_{\lambda-1}(-h_\lambda(x))))) \\ &= \dots = -h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(x))))) = -G_\lambda(x), \quad x \in \mathbb{R}. \end{aligned}$$

Clearly it holds $G_\lambda^{(2)} \in C(\mathbb{R})$.

We notice that

$$\begin{aligned} G_\lambda(+\infty) &= h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(+\infty))))) = \\ &= h_1(h_2(h_3(\dots(h_{\lambda-1}(1)))) = G_{\lambda-1}(1), \end{aligned} \quad (8)$$

and

$$\begin{aligned} G_\lambda(-\infty) &= h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(-\infty))))) = h_1(h_2(h_3(\dots(h_{\lambda-1}(-1)))) \\ &= -h_1(h_2(h_3(\dots(h_{\lambda-1}(1)))) = -G_{\lambda-1}(1). \end{aligned} \quad (9)$$

Consequently, it holds

$$-G_{\lambda-1}(1) \leq G_\lambda(x) \leq G_{\lambda-1}(1), \quad \forall x \in \mathbb{R}. \quad (10)$$

Thus, $y = \pm G_{\lambda-1}(1)$ are asymptotes of $G_\lambda(x)$, any $\lambda \in \mathbb{N}$.

Next we act over $(-\infty, 0]$: let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then by convexity of h_2 there we have

$$h_2(\lambda x + \mu y) \leq \lambda h_2(x) + \mu h_2(y), \quad \forall x, y \in (-\infty, 0];$$

and

$$\begin{aligned} h_1(h_2(\lambda x + \mu y)) &\leq h_1(\lambda h_2(x) + \mu h_2(y)) \leq \\ &\leq \lambda(h_1 \circ h_2)(x) + \mu(h_1 \circ h_2)(y), \quad \forall x, y \in (-\infty, 0]. \end{aligned} \quad (11)$$

So that $h_1 \circ h_2$ is convex over $(-\infty, 0]$.

Now we work on $[0, +\infty)$: let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then by concavity of h_2 there we have

$$h_2(\lambda x + \mu y) \geq \lambda h_2(x) + \mu h_2(y), \quad \forall x, y \in [0, +\infty);$$

and

$$h_1(h_2(\lambda x + \mu y)) \geq h_1(\lambda h_2(x) + \mu h_2(y)) \geq$$

$$\lambda (h_1 \circ h_2)(x) + \mu (h_1 \circ h_2)(y), \quad \forall x, y \in [0, +\infty). \quad (12)$$

Thus, $h_1 \circ h_2$ is concave over $[0, +\infty)$.

Therefore $G_2 = h_1 \circ h_2$ is a general sigmoid activation function with asymptotes $y = \pm h_1(1)$, and fulfilling the rest of conditions of Definition 2.1.

Arguing as above $h_2 \circ h_3 : \mathbb{R} \rightarrow [-1, 1]$, fulfills Definition 2.1, and $h_1 \circ h_2 \circ h_3$ does the same with asymptotes $h = \pm h_1(h_2(1))$.

Inductively, we prove that G_λ fulfills Definition 2.1 with asymptotes $y = \pm G_{\lambda-1}(1)$.

We have established the following:

Theorem 2.1. *Let $q, \lambda \in \mathbb{N}$. Then $G_\lambda^{(q)} := h_1^{(q)} \circ h_2^{(q)} \circ h_3^{(q)} \circ \dots \circ h_\lambda^{(q)}$ fulfill all the properties of Definition 2.1 with asymptotes $y = \pm G_{\lambda-1}^{(q)}(1)$. That is $G_\lambda^{(q)}$ are multi-composite general sigmoid activation functions from $\mathbb{R} \rightarrow [-1, 1]$.*

Corollary 2.2. $\frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}(1)}$ fulfill Definition 2.1 with asymptotes $y = \pm 1$.

We call

$$\tilde{G}_\lambda^{(q)} := \frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}(1)}. \quad (13)$$

Remark 2.1. Next we consider the functions

$$T_\lambda^{(q)}(x) := \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(x+1) - \tilde{G}_\lambda^{(q)}(x-1) \right) > 0, \quad \forall x \in \mathbb{R}, \quad q, \lambda \in \mathbb{N}. \quad (14)$$

We observe that

$$\begin{aligned} T_\lambda^{(q)}(-x) &= \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(-x+1) - \tilde{G}_\lambda^{(q)}(-x-1) \right) = \\ &= \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(-(x-1)) - \tilde{G}_\lambda^{(q)}(-(x+1)) \right) = \frac{1}{4} \left(-\tilde{G}_\lambda^{(q)}(x-1) + \tilde{G}_\lambda^{(q)}(x+1) \right) = \\ &= \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(x+1) - \tilde{G}_\lambda^{(q)}(x-1) \right) = T_\lambda^{(q)}(x). \end{aligned} \quad (15)$$

That is $T_\lambda^{(q)}$ are even functions,

$$T_\lambda^{(q)}(-x) = T_\lambda^{(q)}(x), \quad \forall x \in \mathbb{R}, \quad q, \lambda \in \mathbb{N}. \quad (16)$$

We see that

$$T_\lambda^{(q)}(0) = \frac{\tilde{G}_\lambda^{(q)}(1)}{2}. \quad (17)$$

Let $x > 1$, we have that

$$T_\lambda^{(q)'}(x) = \frac{1}{4} \left(\tilde{G}_\lambda^{(q)'}(x+1) - \tilde{G}_\lambda^{(q)'}(x-1) \right) < 0,$$

by $\tilde{G}_\lambda^{(q)'}$ being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1-x > 0$ and $0 < 1-x < 1+x$. It holds $\tilde{G}_\lambda^{(q)'}(x-1) = \tilde{G}_\lambda^{(q)'}(1-x) > \tilde{G}_\lambda^{(q)'}(x+1)$, so that again $T_\lambda^{(q)'}(x) < 0$. Consequently $T_\lambda^{(q)}$ is strictly decreasing on $(0, +\infty)$.

Clearly, $T_\lambda^{(q)}$ is strictly increasing on $(-\infty, 0)$, and $T_\lambda^{(q)'}(0) = 0$.

Observe that

$$\lim_{x \rightarrow +\infty} T_{\lambda}^{(q)}(x) = \frac{1}{4} \left(\tilde{G}_{\lambda}^{(q)}(+\infty) - \tilde{G}_{\lambda}^{(q)}(+\infty) \right) = 0, \quad (18)$$

and

$$\lim_{x \rightarrow -\infty} T_{\lambda}^{(q)}(x) = \frac{1}{4} \left(\tilde{G}_{\lambda}^{(q)}(-\infty) - \tilde{G}_{\lambda}^{(q)}(-\infty) \right) = 0, \quad q, \lambda \in \mathbb{N}. \quad (19)$$

That is the x -axis is the horizontal asymptote on $T_{\lambda}^{(q)}$.

As a result $T_{\lambda}^{(q)}$ is a bell shaped symmetric function with maximum

$$T_{\lambda}^{(q)}(0) = \frac{\tilde{G}_{\lambda}^{(q)}(1)}{2}. \quad (20)$$

We need

Theorem 2.3. *It holds*

$$\sum_{i=-\infty}^{\infty} T_{\lambda}^{(q)}(x-i) = 1, \quad \forall x \in \mathbb{R}; \quad q, \lambda \in \mathbb{N}. \quad (21)$$

Proof. As similar to [4], p. 286 is omitted. $\square$

Theorem 2.4. *We have that*

$$\int_{-\infty}^{\infty} T_{\lambda}^{(q)}(x) dx = 1, \quad q, \lambda \in \mathbb{N}. \quad (22)$$

Proof. As similar to [4], p. 287, it is omitted. $\square$

So that $T_{\lambda}^{(q)}(x)$ can serve as a density function in general.

We need

Theorem 2.5. *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} T_{\lambda}^{(q)}(nx-k) < \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right), \quad (23)$$

and

$$\lim_{n \rightarrow +\infty} \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) = 0. \quad (24)$$

Proof. As in [4], pp. 531-532, it is omitted. $\square$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We also need

Theorem 2.6. *Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k)} < \frac{1}{T_{\lambda}^{(q)}(1)}, \quad \forall x \in [a, b]; \quad q, \lambda \in \mathbb{N}. \quad (25)$$

Proof. As similar to [4], p. 289 is omitted. $\square$

Remark 2.2. We have that

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k) \neq 1; \quad q, \lambda \in \mathbb{N}, \quad (26)$$

for at least some $x \in [a, b]$.

See [4], p. 290, same reasoning.

Note 2.7. For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (21))

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k) \leq 1. \quad (27)$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 2.2. Let $f \in C([a, b], X)$ and $q, \lambda, n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X -valued linear neural network operators

$${}_{\lambda}^{(q)}K_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) T_{\lambda}^{(q)}(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k)}, \quad x \in [a, b]. \quad (28)$$

Clearly here ${}_{\lambda}^{(q)}K_n(f, x) \in C([a, b], X)$. For convenience we use the same ${}_{\lambda}^{(q)}K_n$ for real valued function when needed. We study here the pointwise and uniform convergence of ${}_{\lambda}^{(q)}K_n(f, x)$ to $f(x)$ with rates.

For convenience also we call

$${}_{\lambda}^{(q)}K_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) T_{\lambda}^{(q)}(nx - k), \quad (29)$$

(similarly ${}_{\lambda}^{(q)}K_n^*$ can be defined for real valued function) that is

$${}_{\lambda}^{(q)}K_n(f, x) = \frac{{}_{\lambda}^{(q)}K_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k)}. \quad (30)$$

So that

$$\begin{aligned} {}_{\lambda}^{(q)}K_n(f, x) - f(x) &= \frac{{}_{\lambda}^{(q)}K_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k)} - f(x) \\ &= \frac{{}_{\lambda}^{(q)}K_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k) \right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k)}. \end{aligned} \quad (31)$$

Consequently we derive

$$\left\| {}_{\lambda}^{(q)}K_n(f, x) - f(x) \right\| \leq \frac{1}{T_{\lambda}^{(q)}(1)} \left\| {}_{\lambda}^{(q)}K_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k) \right) \right\|. \quad (32)$$

That is

$$\left\| {}^{(q)}K_n(f, x) - f(x) \right\| \leq \frac{1}{T_\lambda^{(q)}(1)} \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) T_\lambda^{(q)}(nx - k) \right\|. \quad (33)$$

We will estimate the right hand side of (33).

For that we need, for $f \in C([a, b], X)$ the first modulus of continuity

$$\omega_1(f, \delta)_{[a, b]} := \omega_1(f, \delta) := \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \quad (34)$$

The fact $f \in C([a, b], X)$ is equivalent to $\lim_{\delta \rightarrow 0} \omega_1(f, \delta) = 0$, see [2], p. 154.

We denote by $\|f\|_\infty := \sup_{x \in [a, b]} \|f(x)\|$, for $f \in C([a, b], X)$.

Next we present results of X -valued neural network multicomposite approximation to a function given with rates.

We start with

Theorem 2.8. *Let $f \in C([a, b], X)$, $0 < \alpha < 1$, $q, \lambda, n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a, b]$. Then*

i)

$$\left\| {}^{(q)}K_n(f, x) - f(x) \right\| \leq \frac{1}{T_\lambda^{(q)}(1)} \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2 \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right) \|f\|_\infty \right] =: \Delta_\lambda^{(q)}, \quad (35)$$

and

ii)

$$\left\| {}^{(q)}K_n(f) - f \right\|_\infty \leq \Delta_\lambda^{(q)}. \quad (36)$$

We notice $\lim_{n \rightarrow \infty} {}^{(q)}K_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right)\right)$.

Proof. As similar to [2], p. 293 is omitted. $\square$

For X -valued functions on $[a, b]$ the derivative is defined the same way as the numerical derivative, see [12].

In the next we discuss high order neural network X -valued multicomposite approximation by using the smoothness of f .

Theorem 2.9. *Let $f \in C^N([a, b], X)$, $q, \lambda, n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a, b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\left\| {}^{(q)}K_n(f, x) - f(x) \right\| \leq \frac{1}{T_\lambda^{(q)}(1)} \left\{ \sum_{j=1}^N \frac{\|f^{(j)}(x)\|}{j!} \left[\frac{1}{n^{\alpha j}} + \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right) (b-a)^j \right] + \right.$$

$$\left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \left(1 - \tilde{G}_\lambda^{(q)} (n^{1-\alpha} - 2) \right) \|f^{(N)}\|_\infty (b-a)^N}{N!} \right] \Bigg\} \quad (37)$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$, for some $x_0 \in [a, b]$, it holds

$$\left\| {}^{(q)}_K K_n(f, x_0) - f(x_0) \right\| \leq \frac{1}{T_\lambda^{(q)}(1)}$$

$$\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \left(1 - \tilde{G}_\lambda^{(q)} (n^{1-\alpha} - 2) \right) \|f^{(N)}\|_\infty (b-a)^N}{N!} \right\}, \quad (38)$$

and
iii)

$$\begin{aligned} & \left\| {}^{(q)}_K K_n(f) - f \right\|_\infty \leq \\ & \frac{1}{T_\lambda^{(q)}(1)} \left\{ \sum_{j=1}^N \frac{\|f^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + \left(1 - \tilde{G}_\lambda^{(q)} (n^{1-\alpha} - 2) \right) (b-a)^j \right] + \right. \\ & \left. \left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \left(1 - \tilde{G}_\lambda^{(q)} (n^{1-\alpha} - 2) \right) \|f^{(N)}\|_\infty (b-a)^N}{N!} \right] \right\}. \quad (39) \end{aligned}$$

Again we obtain $\lim_{n \rightarrow \infty} {}^{(q)}_K K_n(f) = f$, pointwise and uniformly.

Proof. As similar to [2], pp. 296-301 is omitted. $\square$

All integrals from now on are of Bochner type [10].

We need

Definition 2.3. ([2, p. 10]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, $f : [a, b] \rightarrow X$. We assume that $f^{(m)} \in L_1([a, b], X)$. We call the Caputo-Bochner left fractional derivative of order α :

$$(D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad \forall x \in [a, b]. \quad (40)$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary X -valued derivative (defined similar to numerical one, see [12], p. 83), and also set $D_{*a}^0 f := f$.

By [2, p. 11], $(D_{*a}^\alpha f)(x)$ exists almost everywhere in $x \in [a, b]$ and $D_{*a}^\alpha f \in L_1([a, b], X)$.

If $\|f^{(m)}\|_{L_\infty([a, b], X)} < \infty$, then by [2, p. 11], $D_{*a}^\alpha f \in C([a, b], X)$, hence $\|D_{*a}^\alpha f\| \in C([a, b])$.

We mention

Lemma 2.10. ([2, p. 150]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a, b], X)$ and $f^{(m)} \in L_\infty([a, b], X)$. Then $D_{*a}^\alpha f(a) = 0$.

We mention

Definition 2.4. ([2, p. 42]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := [\alpha]$. We assume that $f^{(m)} \in L_1([a, b], X)$, where $f : [a, b] \rightarrow X$. We call the Caputo-Bochner right fractional derivative of order α :

$$(D_{b-}^\alpha f)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall x \in [a, b]. \quad (41)$$

We observe that $(D_{b-}^m f)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $(D_{b-}^0 f)(x) = f(x)$.

By [2, p. 43], $(D_{b-}^\alpha f)(x)$ exists almost everywhere on $[a, b]$ and $(D_{b-}^\alpha f) \in L_1([a, b], X)$.

If $\|f^{(m)}\|_{L_\infty([a, b], X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [2, p. 43], $D_{b-}^\alpha f \in C([a, b], X)$, hence $\|D_{b-}^\alpha f\| \in C([a, b])$.

We need

Lemma 2.11. [2, p. 151]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_\infty([a, b], X)$, $m = [\alpha]$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^\alpha f(b) = 0$.

Convention 2.12. We assume that

$$D_{*x_0}^\alpha f(x) = 0, \text{ for } x < x_0, \quad (42)$$

and

$$D_{x_0-}^\alpha f(x) = 0, \text{ for } x > x_0, \quad (43)$$

for all $x, x_0 \in [a, b]$.

We mention

Proposition 2.13. ([2, Ch. 5]) Let $f \in C^n([a, b], X)$, $n = [\nu]$, $\nu > 0$. Then $D_{*a}^\nu f(x)$ is continuous in $x \in [a, b]$.

Proposition 2.14. ([2, Ch. 5]) Let $f \in C^m([a, b], X)$, $m = [\alpha]$, $\alpha > 0$. Then $D_{b-}^\alpha f(x)$ is continuous in $x \in [a, b]$.

We also mention

Proposition 2.15. ([2, Ch. 5]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_\infty([a, b], X)$, $m = [\alpha]$, $\alpha > 0$ and

$$D_{*x_0}^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad (44)$$

for all $x, x_0 \in [a, b] : x \geq x_0$.

Then $D_{*x_0}^\alpha f(x)$ is continuous in x_0 .

Proposition 2.16. ([2, Ch. 5]) Let $f \in C^{m-1}([a, b], X)$, $f^{(m)} \in L_\infty([a, b], X)$, $m = [\alpha]$, $\alpha > 0$ and

$$D_{x_0-}^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta-x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta, \quad (45)$$

for all $x, x_0 \in [a, b] : x_0 \geq x$.

Then $D_{x_0-}^\alpha f(x)$ is continuous in x_0 .

Corollary 2.17. ([2, Ch. 5]) *Let $f \in C^m([a, b], X)$, $m = [\alpha]$, $\alpha > 0$, $x, x_0 \in [a, b]$. Then $D_{*x_0}^\alpha f(x)$, $D_{x_0-}^\alpha f(x)$ are jointly continuous functions in (x, x_0) from $[a, b]^2$ into X , X is a Banach space.*

We need

Theorem 2.18. ([2, Ch. 5]) *Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach space. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x, b]), \quad (46)$$

$\delta > 0$, $x \in [a, b]$.

Then G is continuous on $[a, b]$.

Theorem 2.19. ([2, Ch. 5]) *Let $f : [a, b]^2 \rightarrow X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta, [a, x]), \quad (47)$$

$x \in [a, b]$, *is continuous in $x \in [a, b]$, $\delta > 0$.*

We present the following X -valued fractional approximation result by multicomposed neural networks:

Theorem 2.20. *Let $\alpha > 0$, $N = [\alpha]$, $\alpha \notin \mathbb{N}$, $f \in C^N([a, b], X)$, $0 < \beta < 1$, $x \in [a, b]$, $q, \lambda, n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left\| {}^{(q)}_{\lambda} K_n(f, x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} {}^{(q)}_{\lambda} K_n((\cdot - x)^j)(x) - f(x) \right\| \leq \frac{\left(T_{\lambda}^{(q)}(1) \right)^{-1}}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}})_{[a, x]} + \omega_1(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}})_{[x, b]} \right)}{n^{\alpha\beta}} + \right. \\ \left. (1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\beta} - 2)) \left(\|D_{x-}^{\alpha} f\|_{\infty, [a, x]} (x - a)^{\alpha} + \|D_{*x}^{\alpha} f\|_{\infty, [x, b]} (b - x)^{\alpha} \right) \right\}, \quad (48)$$

ii) if $f^{(j)}(x) = 0$, for $j = 1, \dots, N - 1$, we have

$$\left\| {}^{(q)}_{\lambda} K_n(f, x) - f(x) \right\| \leq \frac{\left(T_{\lambda}^{(q)}(1) \right)^{-1}}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}})_{[a, x]} + \omega_1(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}})_{[x, b]} \right)}{n^{\alpha\beta}} + \right. \\ \left. (1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\beta} - 2)) \left(\|D_{x-}^{\alpha} f\|_{\infty, [a, x]} (x - a)^{\alpha} + \|D_{*x}^{\alpha} f\|_{\infty, [x, b]} (b - x)^{\alpha} \right) \right\}, \quad (49)$$

iii)

$$\left\| {}^{(q)}_{\lambda} K_n(f, x) - f(x) \right\| \leq \left(T_{\lambda}^{(q)}(1) \right)^{-1} \left\{ \sum_{j=1}^{N-1} \frac{\|f^{(j)}(x)\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b - a)^j (1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\beta} - 2)) \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \\ \left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \left(\|D_{x-}^\alpha f\|_{\infty,[a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (50)$$

$\forall x \in [a, b],$
and
iv)

$$\left\| {}^{(q)}_K K_n f - f \right\|_\infty \leq \left(T_\lambda^{(q)}(1) \right)^{-1} \\ \left\{ \sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \right\} + \right. \\ \left. \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \right. \\ \left. \left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) (b-a)^\alpha \left(\sup_{x \in [a,b]} \|D_{x-}^\alpha f\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \|D_{*x}^\alpha f\|_{\infty,[x,b]} \right) \right\} \right\}. \quad (51)$$

Above, when $N = 1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain X -valued fractionally type pointwise and uniform convergence with rates of ${}^{(q)}_K K_n \rightarrow I$ the unit operator, as $n \rightarrow \infty$.

Proof. It is very lengthy, as similar to [2], pp. 305-316, is omitted. $\square$

Next we apply Theorem 2.20 for $N = 1$.

Theorem 2.21. Let $0 < \alpha, \beta < 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $q, \lambda, n \in \mathbb{N} : n^{1-\beta} > 2$. Then

i)

$$\left\| {}^{(q)}_K K_n(f, x) - f(x) \right\| \leq \\ \frac{\left(T_\lambda^{(q)}(1) \right)^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \\ \left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \left(\|D_{x-}^\alpha f\|_{\infty,[a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (52)$$

and

ii)

$$\left\| {}^{(q)}_K K_n f - f \right\|_\infty \leq \frac{\left(T_\lambda^{(q)}(1) \right)^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) (b-a)^\alpha \left(\sup_{x \in [a,b]} \|D_{x-}^\alpha f\|_{\infty, [a,x]} + \sup_{x \in [a,b]} \|D_{*x}^\alpha f\|_{\infty, [x,b]} \right) \right\}. \quad (53)$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 2.22. *Let $0 < \beta < 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $q, \lambda, n \in \mathbb{N}$: $n^{1-\beta} > 2$. Then*

$$\begin{aligned} & i) \quad \left\| {}_\lambda^{(q)} K_n(f, x) - f(x) \right\| \leq \\ & \frac{2 \left(T_\lambda^{(q)}(1) \right)^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right. \\ & \left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \left(\left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty, [a,x]} \sqrt{(x-a)} + \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty, [x,b]} \sqrt{(b-x)} \right) \right\}, \quad (54) \end{aligned}$$

and
ii)

$$\begin{aligned} & \left\| {}_\lambda^{(q)} K_n f - f \right\|_\infty \leq \frac{2 \left(T_\lambda^{(q)}(1) \right)^{-1}}{\sqrt{\pi}} \\ & \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right. \\ & \left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \sqrt{(b-a)} \left(\sup_{x \in [a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty, [a,x]} + \sup_{x \in [a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty, [x,b]} \right) \right\} < \infty. \quad (55) \end{aligned}$$

3. FRACTIONAL CALCULUS WITHIN A PAIR OF BANACH SPACES

We make

Remark 3.1. Throughout this work let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be Banach spaces. Here X^j denotes the j -fold product space $\underbrace{X \times X \times \dots \times X}_j$ endowed with the max-norm $\|x\|_{X^j} := \max_{1 \leq \lambda \leq j} \|x_\lambda\|_1$, where $x := (x_1, \dots, x_j) \in X^j$.

Let the space of $L_j := L_j(X^j, Y)$ of all j -multilinear continuous maps $h : X^j \rightarrow Y$, $j = 1, \dots, m$, which is a Banach space with norm

$$\|h\| = \|h\|_{L_j} := \sup_{(\|x\|_{X^j}=1)} \|h(x)\|_2 = \sup \frac{\|h(x)\|_2}{\|x_1\|_1 \dots \|x_j\|_1}. \quad (56)$$

Let M be a non-empty convex and compact subset of X and $x_0 \in M$ is fixed.

Let O be an open subset of X : $M \subset O$.

Let $f : O \rightarrow Y$ be a continuous function, whose Fréchet derivatives ([15], pp. 87-127) $f^{(j)} : O \rightarrow L_j = L_j(X^j, Y)$ exist and are continuous for $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, \dots, x - x_0) \in X^j$, $x \in M$.

We will work with $f|_M$.

Then, by Taylor's formula ([15], p. 124) we get

$$f(x) = \sum_{j=0}^{m-1} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad (57)$$

for all $x \in M$,

where the remainder $R_m(x, x_0)$ is the Riemann integral

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x - x_0)) \right) (x - x_0)^m du, \quad (58)$$

where we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We obtain (by (56))

$$\left\| \left(f^{(m)}(x_0 + u(x - x_0)) \right) (x - x_0)^m \right\|_2 \leq \left\| f^{(m)}(x_0 + u(x - x_0)) \right\| \|x - x_0\|_1^m. \quad (59)$$

Above $(f|_M)^{(j)}$, $j = 0, 1, \dots, m$, are norm bounded by continuity, and thus they are integrable.

By Corollary 20.3, p. 125, of [15] we obtain

$$\|R_m(x, x_0)\|_2 \leq \sup_{\bar{x} \in L(x_0, x_1)} \left\| f^{(m)}(\bar{x}) \right\|_2 \frac{\|x_1 - x_0\|_1^m}{m!}, \quad (60)$$

where

$$L(x_0, x_1) := \{\bar{x} | \bar{x} = \theta x_1 + (1 - \theta)x_0, 0 \leq \theta \leq 1\} \quad (61)$$

is the line segment joining the points x_0 and x_1 (notice that the last $\bar{x} = x_0 + \theta(x_1 - x_0)$).

Denote also $L(x_0, x_1) = \overline{x_0 x_1}$.

Here $(\cdot - x_0)^j$ maps M into X^j and it is continuous, also $f^{(j)}(x_0)$ maps X^j into Y and it is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into Y .

Let us restrict f on the line segment $\overline{x_0 x_1}$. Then for

$$\bar{x}(u) = ux_1 + (1 - u)x_0 = x_0 + u(x_1 - x_0), \quad 0 \leq u \leq 1,$$

the abstract function

$$f_{\overline{x_0 x_1}}(u) := f(\bar{x}(u)) = f(x_0 + u(x_1 - x_0))$$

will map $[0, 1]$ into an abstract arc in Y , which starts at $y_0 = f(x_0)$ and ends at $y_1 = f(x_1)$.

By [15], p. 124, we have that

$$f_{\overline{x_0x_1}}^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k, \quad (62)$$

for $k = 1, 2, \dots, m$; $u \in [0, 1]$, all continuous in u .

Notice that $f_{\overline{x_0x_1}}^{(m)}$ is also integrable by (59).

We need

Definition 3.1. All as in Remark 3.1. We define the vector left Caputo-Fréchet fractional derivative of $f_{\overline{x_0x_1}}$, of order $\alpha > 0$, $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), by

$$\begin{aligned} D_{*0}^\alpha f_{\overline{x_0x_1}}(u) &= D_{*0}^\alpha (f(x_0 + u(x_1 - x_0))) := \\ J_0^{m-\alpha} \left(f^{(m)}(x_0 + u(x_1 - x_0)) \right) (x_1 - x_0)^m &:= \\ \frac{1}{\Gamma(m-\alpha)} \int_0^u (u-t)^{m-\alpha-1} \left(f^{(m)}(x_0 + t(x_1 - x_0)) \right) (x_1 - x_0)^m dt, \end{aligned} \quad (63)$$

all $0 \leq u \leq 1$,

defined via the vector left Riemann-Liouville fractional integral ([13], p. 2).

Then, we observe that

$$\begin{aligned} J_0^\alpha D_{*0}^\alpha (f(x_0 + u(x_1 - x_0))) &= J_0^\alpha J_0^{m-\alpha} \left(f^{(m)}(x_0 + u(x_1 - x_0)) \right) (x_1 - x_0)^m = \\ (\text{by [13], p. 6}) \end{aligned}$$

$$\begin{aligned} J_0^m \left(f^{(m)}(x_0 + u(x_1 - x_0)) \right) (x_1 - x_0)^m &= \\ \frac{1}{(m-1)!} \int_0^u (u-t)^{m-1} \left(f^{(m)}(x_0 + t(x_1 - x_0)) \right) (x_1 - x_0)^m dt, \end{aligned} \quad (64)$$

true for $0 \leq u \leq 1$.

So, we have proved that

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^u (u-t)^{\alpha-1} D_{*0}^\alpha (f(x_0 + t(x_1 - x_0))) dt &= \\ \frac{1}{(m-1)!} \int_0^u (u-t)^{m-1} \left(f^{(m)}(x_0 + t(x_1 - x_0)) \right) (x_1 - x_0)^m dt, \end{aligned} \quad (65)$$

for all $0 \leq u \leq 1$.

Consequently (by [13], p. 12) it holds the new left fractional-Fréchet Taylor formula on $\overline{x_0x_1}$:

Theorem 3.1. All as above. Then

$$\begin{aligned} f_{\overline{x_0x_1}}(u) &= f(x_0 + u(x_1 - x_0)) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)(x_1 - x_0)^k}{k!} u^k + \\ \frac{1}{\Gamma(\alpha)} \int_0^u (u-t)^{\alpha-1} D_{*0}^\alpha f_{\overline{x_0x_1}}(t) dt, \end{aligned} \quad (66)$$

for all $0 \leq u \leq 1$.

In particular we notice that (the case of $u = 1$)

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} D_{*0}^\alpha (f(x_0 + w(x_1 - x_0))) dw = \quad (67)$$

$$\frac{1}{(m-1)!} \int_0^1 (1-w)^{m-1} \left(f^{(m)}(x_0 + w(x_1 - x_0)) \right) (x_1 - x_0)^m dw = R_m(x_1, x_0),$$

where $R_m(x_1, x_0)$ is as in (58).

So, we have the particular vector left hand side Caputo-Fréchet fractional Taylor's formula ($u = 1$).

Corollary 3.2. *All as above. Then*

$$f(x_1) = \sum_{j=0}^{m-1} \frac{f^{(j)}(x_0)(x_1 - x_0)^j}{j!} + \quad (68)$$

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} D_{*0}^\alpha (f(x_0 + w(x_1 - x_0))) dw,$$

for all $x_0, x_1 \in M$.

We make

Remark 3.2. We are again working on the segment $\overline{x_0 x_1}$, where $x_0, x_1 \in M$; $0 \leq u \leq 1$, with $\bar{x} = \bar{x}(u) := x_0 + u(x_1 - x_0)$, and $f_{\overline{x_0 x_1}}(u) = f(\bar{x}(u)) = f(x_0 + u(x_1 - x_0))$, i.e. $f_{\overline{x_0 x_1}} : [0, 1] \rightarrow Y$.

When $u = 0$, $f_{\overline{x_0 x_1}}(0) = f(x_0)$, and when $u = 1$, $f_{\overline{x_0 x_1}}(1) = f(x_1)$.

We have (by [15], pp. 121-122) the abstract Riemann integral:

$$\int_{x_0}^{x_1} f(\bar{x}) d\bar{x} = \int_0^1 f_{\overline{x_0 x_1}}(u) du. \quad (69)$$

We also have that ([15], p. 122, and p. 124)

$$f'_{\overline{x_0 x_1}}(u) = f'(x_0 + u(x_1 - x_0))(x_1 - x_0),$$

and

$$f_{\overline{x_0 x_1}}^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k, \quad (70)$$

$k = 1, 2, \dots, m$.

Let $\alpha > 0$, such that $\lceil \alpha \rceil = m$.

We consider the vector valued right hand side Riemann-Liouville fractional integral ([13], p. 34),

$$J_{1-\overline{x_0 x_1}}^\alpha(u) = \frac{1}{\Gamma(\alpha)} \int_u^1 (J-u)^{\alpha-1} f_{\overline{x_0 x_1}}(J) dJ, \quad (71)$$

and the vector valued right hand side Caputo fractional derivative of order $\alpha > 0$ of $f_{\overline{x_0 x_1}}$ ([13], p. 42), by

$$D_{1-\overline{x_0 x_1}}^\alpha(u) := (-1)^m J_{1-\overline{x_0 x_1}}^{m-\alpha} f_{\overline{x_0 x_1}}^{(m)}(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_u^1 (J-u)^{m-\alpha-1} f_{\overline{x_0 x_1}}^{(m)}(J) dJ. \quad (72)$$

(we have that $J_{1-}^\alpha J_{-1}^\beta f_{\overline{x_0 x_1}} = J_{-1}^{\alpha+\beta} f_{\overline{x_0 x_1}} = J_{-1}^\beta J_{1-}^\alpha f_{\overline{x_0 x_1}}$, when $f_{\overline{x_0 x_1}} \in C([0, 1], Y)$ or $\alpha + \beta \geq 1$, see [13], p. 39).

We need right hand side Taylor's fractional formula.

Theorem 3.3. ([13], p. 44, by Theorem 2.16) *Let $f \in C^m([0, 1], Y)$, $u \in [0, 1]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Then*

$$f(u) = \sum_{k=0}^{m-1} \frac{f^{(k)}(1)}{k!} (u-1)^k + \frac{1}{\Gamma(\alpha)} \int_u^1 (J-u)^{\alpha-1} D_{1-}^\alpha f(J) dJ. \quad (73)$$

Equation (73) implies the vector right hand side corresponding fractional Taylor's formula:

Theorem 3.4. *All as in Remarks 3.1, 3.2. Then*

$$\begin{aligned} f_{\overline{x_0 x_1}}(u) &= f(x_0 + u(x_1 - x_0)) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1 - x_0)^k}{k!} (u-1)^k + \\ &\quad \frac{1}{\Gamma(\alpha)} \int_u^1 (J-u)^{\alpha-1} D_{1-}^\alpha f_{\overline{x_0 x_1}}(J) dJ, \end{aligned} \quad (74)$$

all $0 \leq u \leq 1$,
where

$$\begin{aligned} D_{1-}^\alpha f_{\overline{x_0 x_1}}(J) &= D_{1-}^\alpha f(x_0 + J(x_1 - x_0)) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_J^1 (t-J)^{m-\alpha-1} f_{\overline{x_0 x_1}}^{(m)}(t) dt \\ &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_J^1 (t-J)^{m-\alpha-1} f^{(m)}(x_0 + t(x_1 - x_0)) (x_1 - x_0)^m dt. \end{aligned} \quad (75)$$

When $u = 0$ we obtain

Corollary 3.5. *All as in Remarks 3.1, 3.2. Then*

$$\begin{aligned} f(x_0) &= \sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1 - x_0)^k}{k!} (-1)^k + \\ &\quad \frac{1}{\Gamma(\alpha)} \int_0^1 J^{\alpha-1} D_{1-}^\alpha f(x_0 + J(x_1 - x_0)) dJ, \end{aligned} \quad (76)$$

for all $x_0, x_1 \in M$.

4. MAIN RESULTS

Again, here all will be as in Sections 2, 3, specializing Section 2 to $[a, b] = [0, 1]$.

So we have the two Banach spaces $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$, M is a non-empty convex and compact subset of X and let $x_0, x_1 \in M$ be fixed points. Let O be an open subset of $X : M \subset O$. Let $f : O \rightarrow Y$ be a continuous function, whose Fréchet derivatives $f^{(j)} : O \rightarrow L_j = L_j(X^j, Y)$ (the space of all j -multilinear maps $h : X^j \rightarrow Y$, $j = 1, \dots, m$) exist and are continuous for $1 \leq j \leq m$, $m \in \mathbb{N}$.

We consider the line segment $\overline{x_0 x_1}$, whose points are described by

$$M \ni \bar{x}(u) := ux_1 + (1-u)x_0 = x_0 + u(x_1 - x_0), \quad 0 \leq u \leq 1, \quad (77)$$

and the abstract function

$$f_{\overline{x_0x_1}}(u) := f(\overline{x}(u)) = f(x_0 + u(x_1 - x_0)), \quad (78)$$

which is continuous in $u \in [0, 1]$.

The last function maps $[0, 1]$ onto an abstract arc in Y , with endpoints $f(x_0)$ and $f(x_1)$. As in (62) we have that

$$f_{\overline{x_0x_1}}^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k, \quad (79)$$

for $k = 1, 2, \dots, m$; $u \in [0, 1]$, all are continuous in u .

Additionally we have that

$$f_{\overline{x_0x_1}}^{(m)}(u) = f^{(m)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^m \quad (80)$$

is integrable on $[0, 1]$.

The left Caputo-Fréchet fractional derivative $D_{*0}^\alpha f_{\overline{x_0x_1}}(u)$ is defined by (63), and the right Caputo-Fréchet fractional derivative $D_{1-}^\alpha f_{\overline{x_0x_1}}(u)$ is defined by (72), where $\alpha > 0$, $\lceil \alpha \rceil = m$.

By Definition 2.2, for $f_{\overline{x_0x_1}} \in C([0, 1], Y)$ and $q, \lambda, n \in \mathbb{N}$, we have that

$${}_{\lambda}^{(q)}K_n(f_{\overline{x_0x_1}}, u) := \frac{\sum_{k=0}^n f_{\overline{x_0x_1}}\left(\frac{k}{n}\right) T_{\lambda}^{(q)}(nu - k)}{\sum_{k=0}^n T_{\lambda}^{(q)}(nu - k)}, \quad u \in [0, 1], \quad (81)$$

of course ${}_{\lambda}^{(q)}K_n(f_{\overline{x_0x_1}}, u) \in C([0, 1], Y)$.

Next, we give results for the pointwise and uniform convergence of ${}_{\lambda}^{(q)}K_n(f_{\overline{x_0x_1}}, u)$ to $f_{\overline{x_0x_1}}(u)$ with rates.

A general result follows:

Theorem 4.1. *Let $0 < \alpha < 1$, $q, \lambda, n \in \mathbb{N} : n^{1-\alpha} > 2$, $u \in [0, 1]$. Then*

i)

$$\left\| {}_{\lambda}^{(q)}K_n(f_{\overline{x_0x_1}}, u) - f_{\overline{x_0x_1}}(u) \right\|_2 \leq \quad (82)$$

$$\frac{1}{T_{\lambda}^{(q)}(1)} \left[\omega_1 \left(f_{\overline{x_0x_1}}, \frac{1}{n^\alpha} \right) + 2 \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \|f_{\overline{x_0x_1}}\|_{\infty} \right] =: \delta_{\lambda}^{(q)},$$

and

ii)

$$\left\| {}_{\lambda}^{(q)}K_n(f_{\overline{x_0x_1}}) - f_{\overline{x_0x_1}} \right\|_{\infty} \leq \delta_{\lambda}^{(q)}. \quad (83)$$

We notice that $\lim_{n \rightarrow \infty} {}_{\lambda}^{(q)}K_n(f_{\overline{x_0x_1}}) = f_{\overline{x_0x_1}}$, pointwise and uniformly.

The speed of convergence is $\max \left(\frac{1}{n^\alpha}, \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right)$.

Proof. By Theorem 2.8. □

Using differentiation we derive:

Theorem 4.2. Let $q, \lambda, n, N \in \mathbb{N}$, $0 < \alpha < 1$, $u \in [0, 1]$ and $n^{1-\alpha} > 2$. Then

$$\begin{aligned} i) \quad & \left\| {}^{(q)}_K K_n(f_{\overline{x_0 x_1}}, u) - f_{\overline{x_0 x_1}}(u) \right\|_2 \leq \\ & \frac{1}{T_\lambda^{(q)}(1)} \left\{ \sum_{j=1}^N \frac{\|f_{\overline{x_0 x_1}}^{(j)}(u)\|_2}{j!} \left[\frac{1}{n^{\alpha j}} + \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right) \right] + \right. \\ & \left. \left[\omega_1 \left(f_{\overline{x_0 x_1}}^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right)}{N!} \|f_{\overline{x_0 x_1}}^{(N)}\|_\infty \right] \right\} \end{aligned} \quad (84)$$

ii) assume further $f_{\overline{x_0 x_1}}^{(j)}(t_0) = 0$, $j = 1, \dots, N$, for some $t_0 \in [0, 1]$, it holds

$$\begin{aligned} & \left\| {}^{(q)}_K K_n(f_{\overline{x_0 x_1}}, t_0) - f_{\overline{x_0 x_1}}(t_0) \right\|_2 \leq \frac{1}{T_\lambda^{(q)}(1)} \\ & \left\{ \omega_1 \left(f_{\overline{x_0 x_1}}^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right)}{N!} \|f_{\overline{x_0 x_1}}^{(N)}\|_\infty \right\}, \end{aligned} \quad (85)$$

a much faster convergence,

and

iii)

$$\begin{aligned} & \left\| {}^{(q)}_K K_n(f_{\overline{x_0 x_1}}) - f_{\overline{x_0 x_1}} \right\|_\infty \leq \\ & \frac{1}{T_\lambda^{(q)}(1)} \left\{ \sum_{j=1}^N \frac{\|f_{\overline{x_0 x_1}}^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right) \right] + \right. \\ & \left. \left[\omega_1 \left(f_{\overline{x_0 x_1}}^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right) \|f_{\overline{x_0 x_1}}^{(N)}\|_\infty}{N!} \right] \right\}. \end{aligned} \quad (86)$$

Again, we derive $\lim_{n \rightarrow \infty} {}^{(q)}_K K_n(f_{\overline{x_0 x_1}}) = f_{\overline{x_0 x_1}}$, pointwise and uniformly.

Proof. By Theorem 2.9. □

Using fractional differentiation we obtain:

Theorem 4.3. Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $0 < \beta < 1$, $t \in [0, 1]$, $q, \lambda, n \in \mathbb{N}$: $n^{1-\beta} > 2$. Then

i)

$$\begin{aligned} & \left\| {}^{(q)}_K K_n(f_{\overline{x_0 x_1}}, t) - \sum_{j=1}^{N-1} \frac{f_{\overline{x_0 x_1}}^{(j)}(t)}{j!} {}^{(q)}_K K_n((\cdot - t)^j)(t) - f_{\overline{x_0 x_1}}(t) \right\|_2 \leq \\ & \frac{\left(T_\lambda^{(q)}(1)\right)^{-1}}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1(D_{t-}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta})_{[0, t]} + \omega_1(D_{*t}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta})_{[t, 1]}\right)}{n^{\alpha\beta}} + \right. \end{aligned}$$

$$\left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\beta} - 2)\right) \left(\|D_{t-}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [0, t]} t^\alpha + \|D_{*t}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [t, 1]} (1-t)^\alpha\right)\}, \quad (87)$$

ii) if $f_{\overline{x_0 x_1}}^{(j)}(t) = 0$, for $j = 1, \dots, N-1$, we have

$$\begin{aligned} & \left\| {}^{(q)}K_n(f_{\overline{x_0 x_1}}, t) - f_{\overline{x_0 x_1}}(t) \right\|_2 \leq \frac{\left(T_\lambda^{(q)}(1)\right)^{-1}}{\Gamma(\alpha + 1)} \\ & \left\{ \frac{\left(\omega_1 \left(D_{t-}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta}\right)_{[0, t]} + \omega_1 \left(D_{*t}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta}\right)_{[t, 1]}\right)}{n^{\alpha\beta}} + \right. \\ & \left. \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\beta} - 2)\right) \left(\|D_{t-}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [0, t]} t^\alpha + \|D_{*t}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [t, 1]} (1-t)^\alpha\right) \right\}, \quad (88) \end{aligned}$$

here convergence is at a higher rate,

iii)

$$\begin{aligned} & \left\| {}^{(q)}K_n(f_{\overline{x_0 x_1}}, t) - f_{\overline{x_0 x_1}}(t) \right\|_2 \leq \left(T_\lambda^{(q)}(1)\right)^{-1} \\ & \left\{ \sum_{j=1}^{N-1} \frac{\left\| f_{\overline{x_0 x_1}}^{(j)}(t) \right\|_2}{j!} \left\{ \frac{1}{n^{\beta j}} + \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\beta} - 2)\right) \right\} + \right. \\ & \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta}\right)_{[0, t]} + \omega_1 \left(D_{*t}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta}\right)_{[t, 1]}\right)}{n^{\alpha\beta}} + \right. \\ & \left. \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\beta} - 2)\right) \left(\|D_{t-}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [0, t]} t^\alpha + \|D_{*t}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [t, 1]} (1-t)^\alpha\right) \right\} \Big\}, \quad (89) \end{aligned}$$

$\forall t \in [0, 1]$,

and

iv)

$$\begin{aligned} & \left\| {}^{(q)}K_n f_{\overline{x_0 x_1}} - f_{\overline{x_0 x_1}} \right\|_\infty \leq \left(T_\lambda^{(q)}(1)\right)^{-1} \\ & \left\{ \sum_{j=1}^{N-1} \frac{\left\| f_{\overline{x_0 x_1}}^{(j)} \right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\beta} - 2)\right) \right\} + \right. \\ & \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\sup_{t \in [0, 1]} \omega_1 \left(D_{t-}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta}\right)_{[0, t]} + \sup_{t \in [0, 1]} \omega_1 \left(D_{*t}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta}\right)_{[t, 1]}\right)}{n^{\alpha\beta}} + \right. \\ & \left. \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\beta} - 2)\right) \left(\sup_{t \in [0, 1]} \|D_{t-}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [0, t]} + \sup_{t \in [0, 1]} \|D_{*t}^\alpha f_{\overline{x_0 x_1}}\|_{\infty, [t, 1]}\right) \right\} \Big\}. \quad (90) \end{aligned}$$

Above, when $N = 1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

We derive the Y -valued fractionally type pointwise and uniform convergence with rates of ${}^{(q)}K_n \rightarrow I$ the unit operator, as $n \rightarrow \infty$.

Proof. By Theorem 2.20. □

Next we apply Theorem 4.3 for $N = 1$.

Theorem 4.4. *Let $0 < \alpha, \beta < 1$, $t \in [0, 1]$, $q, \lambda, n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left\| {}^{(q)}_K K_n (f_{\overline{x_0 x_1}}, t) - f_{\overline{x_0 x_1}}(t) \right\|_2 \leq \quad (91)$$

$$\frac{\left(T_\lambda^{(q)}(1) \right)^{-1}}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[0,t]} + \omega_1 \left(D_{*t}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[t,1]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \left(\| D_{t-}^\alpha f_{\overline{x_0 x_1}} \|_{\infty, [0,t]} t^\alpha + \| D_{*t}^\alpha f_{\overline{x_0 x_1}} \|_{\infty, [t,1]} (1-t)^\alpha \right) \right\},$$

ii)

$$\left\| {}^{(q)}_K K_n f_{\overline{x_0 x_1}} - f_{\overline{x_0 x_1}} \right\|_\infty \leq \frac{\left(T_\lambda^{(q)}(1) \right)^{-1}}{\Gamma(\alpha + 1)}$$

$$\left\{ \frac{\left(\sup_{t \in [0,1]} \omega_1 \left(D_{t-}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[0,t]} + \sup_{t \in [0,1]} \omega_1 \left(D_{*t}^\alpha f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[t,1]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \left(\sup_{t \in [0,1]} \| D_{t-}^\alpha f_{\overline{x_0 x_1}} \|_{\infty, [0,t]} + \sup_{t \in [0,1]} \| D_{*t}^\alpha f_{\overline{x_0 x_1}} \|_{\infty, [t,1]} \right) \right\}. \quad (92)$$

Finally when $\alpha = \frac{1}{2}$ we obtain

Corollary 4.5. *Let $0 < \beta < 1$, $t \in [0, 1]$, $q, \lambda, n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left\| {}^{(q)}_K K_n (f_{\overline{x_0 x_1}}, t) - f_{\overline{x_0 x_1}}(t) \right\|_2 \leq$$

$$\frac{2 \left(T_\lambda^{(q)}(1) \right)^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\frac{1}{2}} f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[0,t]} + \omega_1 \left(D_{*t}^{\frac{1}{2}} f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[t,1]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \left(\left\| D_{t-}^{\frac{1}{2}} f_{\overline{x_0 x_1}} \right\|_{\infty, [0,t]} \sqrt{t} + \left\| D_{*t}^{\frac{1}{2}} f_{\overline{x_0 x_1}} \right\|_{\infty, [t,1]} \sqrt{1-t} \right) \right\}, \quad (93)$$

and

ii)

$$\left\| {}^{(q)}_K K_n f_{\overline{x_0 x_1}} - f_{\overline{x_0 x_1}} \right\|_\infty \leq \frac{2 \left(T_\lambda^{(q)}(1) \right)^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{t \in [0,1]} \omega_1 \left(D_{t-}^{\frac{1}{2}} f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[0,t]} + \sup_{t \in [0,1]} \omega_1 \left(D_{*t}^{\frac{1}{2}} f_{\overline{x_0 x_1}}, \frac{1}{n^\beta} \right)_{[t,1]} \right)}{n^{\frac{\beta}{2}}} + \left(1 - \tilde{G}_\lambda^{(q)} \left(n^{1-\beta} - 2 \right) \right) \left(\sup_{t \in [0,1]} \left\| D_{t-}^{\frac{1}{2}} f_{\overline{x_0 x_1}} \right\|_{\infty, [0,t]} + \sup_{t \in [0,1]} \left\| D_{*t}^{\frac{1}{2}} f_{\overline{x_0 x_1}} \right\|_{\infty, [t,1]} \right) \right\} < \infty. \quad (94)$$

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