

SYMMETRIZED PERTURBED GENERALIZED LOGISTIC UNIVARIATE ORDINARY AND GENERALIZED FRACTIONAL NEURAL NETWORK APPROXIMATION WITH RATES

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ABSTRACT. We explore here the symmetrized neural network operators able to achieve supersonic speeds of convergence. That is based on the following principle: a "symmetric neural network half" refers to a neural network architecture designed to specifically exploit symmetry in data, where only one half of the network needs to be computed due to the mirrored structure, effectively reducing computational cost by roughly half while still capturing the symmetry present in the input data. Furthermore, the brain non-symmetry has been observed in animals and humans in terms of structure, function and behavior. This lateralization is thought to reflect evolutionary, hereditary, developmental, experiential and pathological factors. Consequently, it is natural to consider for our study a deformed neural network activation function and operator, though finally we symmetrize to go faster. Thus, this work is a focused study under this philosophy of approaching reality as closely as possible. The activation function here is the parametrized and deformed generalized logistic function. We perform precise quantitative approximations of univariate ordinary and generalized fractional types. The generalized fractional case is treated for the first time.

1. INTRODUCTION

Symmetrization in neural networks refers to embedding geometric, algebraic or statistical symmetries (such as rotations, translations, or permutations) into a model. This drastically reduces sample complexity, ensures physically or geometrically valid predictions, and stabilizes training.

Key Types of Symmetrization are:

1. Geometric & Algebraic Symmetries (Equivariance/Invariance)

Rather than forcing a neural network to learn the symmetries of your data through massive amounts of training, architectures can be explicitly built to preserve them.

- **Invariance:** The network's output remains the same, regardless of the transformation. (e.g., Identifying a cat in an image, no matter its orientation).

2020 *Mathematics Subject Classification.* 26A33, 41A17, 41A25, 41A30, 46B25.

Key words and phrases. Parametrized and Perturbed generalized logistic function, quasi-interpolation symmetrized neural network operators, univariate ordinary and generalized fractional approximation, quantitative approximations.

- **Equivariance:** The network’s internal representations change in the exact same way as in the input is transformed. (e.g., A pixel movement in an image causes an identical spatial shift in the feature map).
- **Techniques:**
 - **Convolutional Neural Networks (CNNs):** Enforce translation equivariance.
 - **Graph Neural Networks (GNNs) & PointNet:** Use pooling operations (like sum, max, or mean) to enforce permutation invariance (the order of input nodes/points doesn’t matter).
 - **E(n)-Equivariant Neural Networks (EGNNs):** Widely used in chemistry and physics, where molecular properties need to be invariant to 3D rotations and translations.

2. Symmetrization of Loss Functions

In classification problems and label noise learning, loss functions can be ”symmetrized” to make the model robust against corrupted or noisy labels. A symmetric loss function ensures that gradients do not get trapped in local minima caused by noisy data labels, pushing the network to learn genuine patterns rather than overfitting to noise.

Common Applications:

- **Physics & Materials Science:** Enforcing thermodynamic or physical symmetries directly into networks predicting material properties so predictions don’t violate conservation laws.
- **Drug Discovery & Quantum Chemistry:** Predicting molecular interactions while maintaining rotational and translational invariance in 3D space.
- **Computer Vision:** Modelling point clouds and sets where geometric symmetries are strictly defined.

Why Symmetrization Matters:

Enforcing symmetry restricts the network’s hypothesis space to only consider functions that obey those symmetries.

- **Pros:** Highly sample-efficient; guaranteed adherence to physical, logical, or geometric rules; fewer parameters required.
- **Cons/Challenges:** Requires explicit mathematical knowledge of the symmetries in your system; if the wrong symmetry is enforced, the model’s accuracy degrades completely.

The Opposite: ”Symmetry Breaking”

While symmetrization is used to build robust inductive biases, neural network optimization also relies on **symmetry breaking**. If all weights in a hidden layer are initialized to the same value, all neurons would learn the exact same features during gradient descent. Random initialization and techniques like weight perturbation intentionally break this initial symmetry so that individual neurons can learn different, diverse features.

In this work we introduce symmetrized neural network operators that achieve remarkably fast convergence. This is based on a mirrored architecture where only half of the network requires computation. By specifically exploiting data symmetry,

we halve the computational cost while perfectly preserving input symmetries. Additionally, because biological brains exhibit structural, functional, and behavioral assymetries (lateralization) shaped by evolutionary and experiential factors, our approach incorporates a deformed neural network operator and activation function—specifically, a parametrized and deformed generalized logistic function. We then apply symmetry to accelerate the process. This methodology reflects our core philosophy: balancing neurobiological realism with optimized performance. Using this approach, we deliver precise quantitative approximations for both standard univariate and novel generalized fractional cases.

The author has been the pioneer in precise quantitative Neural Network approximation, [2], 1997.

Since then he has published numerous papers and books on the topic to all possible directions including fractional neural network approximation. Most notably he has published the monographs [3], [5]-[7], [9]-[11]. The author has been inspired also by [12]-[14].

Our neural networks here are with one hidden layer.

Real feed-forward neural networks (FNNs) with one hidden layer are mathematically expressed by

$$N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x , and σ is the activation function of the network.

2. ESSENTIALS ON SYMMETRIZED NEURAL NETWORKS

Here we follow [11], pp. 395-417.

Our activation function here to be used is the q -deformed and λ -parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R}, \quad q, \lambda > 0, \quad A > 1. \quad (1)$$

This is the A -generalized logistic function.

For more read Chapter 16 of [11]: "Banach space valued ordinary and fractional neural network approximation based on q -deformed and λ -parametrized A -generalized logistic function".

This chapter motivates our current work.

The proposed "symmetrization technique" aims to use half data feed to our neural networks.

We will employ the following density function

$$G_{q,\lambda}(x) := \frac{1}{2} (\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)), \quad x \in \mathbb{R}, \quad q, \lambda > 0. \quad (2)$$

We have that

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \quad (3)$$

and

$$G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x), \quad \forall x \in \mathbb{R}. \quad (4)$$

Adding (3) and (4) we obtain

$$G_{q,\lambda}(-x) + G_{\frac{1}{q},\lambda}(-x) = G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R}, \quad (5)$$

the key to this work.

So that

$$W(x) := \frac{G_{q,\lambda}(x) + G_{\frac{1}{q},\lambda}(x)}{2} \quad (6)$$

is an even function, symmetric with respect to the y -axis.

The global maximum of $G_{q,\lambda}$ is given by (16.18), p. 401 of [11] as

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2(A^\lambda + 1)}. \quad (7)$$

And, the global max of $G_{\frac{1}{q},\lambda}$ is

$$G_{\frac{1}{q},\lambda}\left(\frac{\log_A \frac{1}{q}}{\lambda}\right) = G_{\frac{1}{q},\lambda}\left(\frac{-\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2(A^\lambda + 1)}, \quad (8)$$

both sharing the same maximum at symmetric points.

By Theorem 16.1, p. 401 of [11], we have that

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x-i) = 1, \quad \forall x \in \mathbb{R}, \lambda, q > 0, A > 1, \quad (9)$$

and

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x-i) = 1, \quad \forall x \in \mathbb{R}, \lambda, q > 0, A > 1. \quad (10)$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} W(x-i) = 1, \quad \forall x \in \mathbb{R}. \quad (11)$$

By Theorem 16.2, p. 402 of [11], we have that

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x) dx = 1, \quad \lambda, q > 0, A > 1, \quad (12)$$

similarly it holds

$$\int_{-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x) dx = 1, \quad (13)$$

so that

$$\int_{-\infty}^{\infty} W(x) dx = 1, \quad (14)$$

therefore $W(x)$ is a density function.

By Theorem 16.3, p. 402 of [11], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then

$$\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(nx - k) < 2 \max \left\{ q, \frac{1}{q} \right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}} = \gamma A^{-\lambda(n^{1-\alpha}-2)}, \quad (15)$$

where $\lambda, q > 0$, $A > 1$; $\gamma := 2 \max \left\{ q, \frac{1}{q} \right\}$.

Similarly, we get that

$$\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\alpha}}}^{\infty} G_{\frac{1}{q},\lambda}(nx - k) < \gamma A^{-\lambda(n^{1-\alpha}-2)}. \quad (16)$$

Consequently we obtain that

$$\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\alpha}}}^{\infty} W(nx - k) < \gamma A^{-\lambda(n^{1-\alpha}-2)}, \quad (17)$$

where $\gamma := 2 \max \left\{ q, \frac{1}{q} \right\}$.

Here $\lceil \cdot \rceil$ denotes the ceiling of the number, and $\lfloor \cdot \rfloor$ its integral part.

We mention

Theorem 2.1. (p. 402, [11]) Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Then

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx - k)} < \max \left\{ \frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}(\lambda_{\frac{1}{q}})} \right\} =: K(q). \quad (18)$$

Similarly, we consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$.

Thus

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx - k)} < \max \left\{ \frac{1}{G_{\frac{1}{q},\lambda}(\lambda_{\frac{1}{q}})}, \frac{1}{G_{q,\lambda}(\lambda_q)} \right\} = K(q). \quad (19)$$

Hence

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx - k) > \frac{1}{K(q)}, \quad (20)$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx - k) > \frac{1}{K(q)}. \quad (21)$$

Consequently it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(G_{q,\lambda}(nx-k) + G_{\frac{1}{q},\lambda}(nx-k) \right)}{2} > \frac{2}{2K(q)} = \frac{1}{K(q)}, \quad (22)$$

so that

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(G_{q,\lambda}(nx-k) + G_{\frac{1}{q},\lambda}(nx-k) \right)}{2}} < K(q), \quad (23)$$

that is

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k)} < K(q). \quad (24)$$

We have proved

Theorem 2.2. *Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q, \lambda > 0$, $A > 1$, we consider $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$, and $\lambda_q > 1$. Also consider $\lambda_{\frac{1}{q}} > z_1 > 0$, such that $G_{\frac{1}{q},\lambda}(z_1) = G_{\frac{1}{q},\lambda}(0)$, and $\lambda_{\frac{1}{q}} > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k)} < K(q). \quad (25)$$

We make

Remark 2.1. I) By Remark 16.5, p. 402 of [11], we have

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx_1 - k) \neq 1, \text{ for some } x_1 \in [a, b], \quad (26)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx_2 - k) \neq 1, \text{ for some } x_2 \in [a, b]. \quad (27)$$

Therefore it holds

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(G_{q,\lambda}(nx_1 - k) + G_{\frac{1}{q},\lambda}(nx_2 - k) \right)}{2} \neq 1. \quad (28)$$

Hence it is

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\left(G_{q,\lambda}(nx_1 - k) + G_{\frac{1}{q},\lambda}(nx_1 - k) \right)}{2} \neq 1, \quad (29)$$

even if

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{\frac{1}{q},\lambda}(nx_1 - k) = 1, \quad (30)$$

because then

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{G_{q,\lambda}(nx_1 - k)}{2} + \frac{1}{2} \neq 1, \quad (31)$$

equivalently

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{G_{q,\lambda}(nx_1 - k)}{2} \neq \frac{1}{2}, \quad (32)$$

true by

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx_1 - k) \neq 1. \quad (33)$$

II) Let $[a, b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. So in general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \leq 1. \quad (34)$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 2.1. Let $f \in C([a, b], X)$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X -valued linear symmetrized neural network operators

$$S_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) W(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k)} = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \left(G_{q,\lambda}(nx_1 - k) + G_{\frac{1}{q},\lambda}(nx_1 - k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(G_{q,\lambda}(nx_1 - k) + G_{\frac{1}{q},\lambda}(nx_1 - k)\right)}. \quad (35)$$

The modulus of continuity is defined by

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \quad (36)$$

The same is defined for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions) and for $f \in C_B(\mathbb{R}, X)$ (bounded and continuous functions) and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous functions).

The fact $f \in C([a, b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \rightarrow 0} \omega_1(f, \delta) = 0$.

Definition 2.2. Let $f \in C_{uB}(\mathbb{R}, X)$ or $f \in C_B(\mathbb{R}, X)$, we define

$$\bar{S}_n(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) W(nx - k), \quad (37)$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$, the X -valued quasi-interpolation symmetrized perturbed neural network operator.

We mention some basic X -valued symmetrized perturbed neural network approximations to a function given quantitatively.

Theorem 2.3. *Let $f \in C([a, b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a, b]$. Then*

i)

$$\|S_n(f, x) - f(x)\| \leq K(q) \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] =: \rho_s \quad (38)$$

and

ii)

$$\|S_n(f) - f\|_\infty \leq \rho_s. \quad (39)$$

We get that $\lim_{n \rightarrow \infty} S_n(f) = f$, pointwise and uniformly.

Proof. Similar to the proof of Theorem 16.9, p. 405, [11]. $\square$

Next we give

Theorem 2.4. *Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then*

i)

$$\|\bar{S}_n(f, x) - f(x)\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty \gamma A^{-\lambda(n^{1-\alpha}-2)} =: \gamma_s \quad (40)$$

and

ii)

$$\|\bar{S}_n f - f\|_\infty \leq \gamma_s. \quad (41)$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n \rightarrow \infty} \bar{S}_n(f) = f$, pointwise and uniformly.

Proof. Similar to Theorem 16.10, p. 406, [11]. $\square$

In the next we discuss high order neural network X -valued approximation by using the smoothness of f .

Theorem 2.5. *Let $f \in C^N([a, b], X)$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a, b]$, $n^{1-\alpha} > 2$. Then*

i)

$$\|S_n(f, x) - f(x)\| \leq K(q) \left\{ \sum_{j=1}^N \frac{\|f^{(j)}(x)\|}{j!} \left[\frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] + \left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{2\|f^{(N)}\|_\infty (b-a)^N \gamma A^{-\lambda(n^{1-\alpha}-2)}}{N!} \right] \right\}, \quad (42)$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$, for $x_0 \in [a, b]$, thus

$$\|S_n(f, x_0) - f(x_0)\| \leq K(q).$$

$$\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \|f^{(N)}\|_\infty (b-a)^N}{N!} \gamma A^{-\lambda(n^{1-\alpha}-2)} \right\}, \quad (43)$$

and
iii)

$$\|S_n(f) - f\|_\infty \leq K(q) \left\{ \sum_{j=1}^N \frac{\|f^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] + \left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \|f^{(N)}\|_\infty (b-a)^N \gamma A^{-\lambda(n^{1-\alpha}-2)}}{N!} \right] \right\}. \quad (44)$$

Again we obtain $\lim_{n \rightarrow \infty} S_n(f) = f$, pointwise and uniform.

Proof. It is lengthy and similar to Theorem 16.12, p. 408 of [11]. As such it is omitted. $\square$

3. SPECIAL SYMMETRIZED NEURAL NETWORK APPROXIMATIONS

Here $X = \mathbb{R}$. From [4], we will be using the following three theorems on extended Taylor formula.

Theorem 3.1. *Let $N \in \mathbb{N}$, $0 < \varepsilon < \frac{\pi}{2}$ small, and $f \in C^N \left(\left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon \right] \right)$; $x, y \in \left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon \right]$. Then*

$$f(x) = f(y) + \sum_{k=1}^N \frac{(f \circ \sin^{-1})^{(k)}(\sin y)}{k!} (\sin x - \sin y)^k + K_N(y, x), \quad (45)$$

where

$$K_N(y, x) = \frac{1}{(N-1)!}. \quad (46)$$

$$\int_y^x (\sin x - \sin s)^{N-1} \left((f \circ \sin^{-1})^{(N)}(\sin s) - (f \circ \sin^{-1})^{(N)}(\sin y) \right) \cos s \, ds.$$

Theorem 3.2. *Let $f \in C^N([\varepsilon, \pi - \varepsilon])$, $N \in \mathbb{N}$, $\varepsilon > 0$ small; $x, y \in [\varepsilon, \pi - \varepsilon]$. Then*

$$f(x) = f(y) + \sum_{k=1}^N \frac{(f \circ \cos^{-1})^{(k)}(\cos y)}{k!} (\cos x - \cos y)^k + K_N^*(y, x), \quad (47)$$

where

$$K_N^*(y, x) = -\frac{1}{(N-1)!}. \quad (48)$$

$$\int_y^x (\cos x - \cos s)^{N-1} \left[(f \circ \cos^{-1})^{(N)}(\cos s) - (f \circ \cos^{-1})^{(N)}(\cos y) \right] \sin s \, ds.$$

Theorem 3.3. Let $f \in C^N([a, b])$ (or $f \in C^N(\mathbb{R})$), $N \in \mathbb{N}$; $x, y \in [a, b]$ (or $x, y \in \mathbb{R}$). Then

$$f(x) = f(y) + \sum_{k=1}^N \frac{\left(f \circ \ln_{\frac{1}{e}}\right)^{(k)}(e^{-y})}{k!} (e^{-x} - e^{-y})^k + \bar{K}_N(y, x), \quad (49)$$

where

$$\bar{K}_N(y, x) = -\frac{1}{(N-1)!}. \quad (50)$$

$$\int_y^x (e^{-x} - e^{-s})^{N-1} \left[\left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}(e^{-s}) - \left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}(e^{-y}) \right] e^{-s} ds.$$

Remark 3.1. Using the mean value theorem we get

$$\begin{aligned} |\sin x - \sin y| &\leq |x - y|, \\ |\cos x - \cos y| &\leq |x - y|, \quad \forall x, y \in \mathbb{R}, \end{aligned} \quad (51)$$

furthermore we have

$$\begin{aligned} |\sin x - \sin y| &\leq 2, \\ |\cos x - \cos y| &\leq 2, \quad \forall x, y \in \mathbb{R}. \end{aligned} \quad (52)$$

Similarly we get

$$|e^{-x} - e^{-y}| \leq e^{-a} |x - y|, \quad (53)$$

and

$$|e^{-x} - e^{-y}| \leq e^{-a} - e^{-b}, \quad \forall x, y \in [a, b]. \quad (54)$$

Let $g(x) = \ln_{\frac{1}{e}} x$, $\sin^{-1} x$, $\cos^{-1} x$ and assume $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$. Then, by [6], p. 401, we get $(f \circ g^{-1})^{(j)}(g(x_0)) = 0$, $j = 1, \dots, N$.

In the next three Theorems 3.4-3.6 we present more general and flexible upper bounds to our error symmetrized quantities.

We give

Theorem 3.4. Let $f \in C^N([a, b])$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a, b]$. Then

1)

$$\begin{aligned} \left| S_n(f, x) - \sum_{j=1}^N \frac{\left(f \circ \ln_{\frac{1}{e}}\right)^{(j)}(e^{-x})}{j!} S_n\left((e^{-\cdot} - e^{-x})^j, x\right) - f(x) \right| \leq \\ K(q) \left[\frac{e^{-\alpha N}}{N! n^{N\alpha}} \omega_1\left(\left(f \circ \ln_{\frac{1}{e}}\right)^{(N)}, \frac{e^{-\alpha}}{n^\alpha}\right) + \right. \\ \left. \frac{2(e^{-a} - e^{-b})}{N!} \left\| \left(f \circ \ln_{\frac{1}{e}}\right)^{(N)} \right\|_\infty \gamma A^{-\lambda(n^{1-\alpha}-2)} \right], \end{aligned} \quad (55)$$

2)

$$|S_n(f, x) - f(x)| \leq K(q). \quad (56)$$

$$\left\{ \sum_{j=1}^N \frac{\left| \left(f \circ \ln_{\frac{1}{e}} \right)^{(j)} (e^{-x}) \right|}{j!} e^{-aj} \left[\frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] + \right. \\ \left. \left[\frac{e^{-aN}}{N! n^{\alpha N}} \omega_1 \left(\left(f \circ \ln_{\frac{1}{e}} \right)^{(N)}, \frac{e^{-a}}{n^{\alpha}} \right) + \frac{2(e^{-a} - e^{-b})^N}{N!} \left\| \left(f \circ \ln_{\frac{1}{e}} \right)^{(N)} \right\|_{\infty} \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] \right\},$$

3) If $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$, it holds

$$|S_n(f, x_0) - f(x_0)| \leq K(q). \quad (57)$$

$$\left[\frac{e^{-aN}}{N! n^{\alpha N}} \omega_1 \left(\left(f \circ \ln_{\frac{1}{e}} \right)^{(N)}, \frac{e^{-a}}{n^{\alpha}} \right) + \frac{2(e^{-a} - e^{-b})^N}{N!} \left\| \left(f \circ \ln_{\frac{1}{e}} \right)^{(N)} \right\|_{\infty} \gamma A^{-\lambda(n^{1-\alpha}-2)} \right].$$

Observe here the speed of convergence is extremely high at $\frac{1}{n^{(N+1)\alpha}}$.

Proof. Call $F := f \circ \ln_{\frac{1}{e}}$. Let $x, \frac{k}{n} \in [a, b]$. Then

$$f\left(\frac{k}{n}\right) - f(x) = \sum_{j=1}^N \frac{F^{(j)}(e^{-x})}{j!} \left(e^{-\frac{k}{n}} - e^{-x}\right)^j + \bar{K}_N\left(x, \frac{k}{n}\right),$$

where

$$\bar{K}_N\left(x, \frac{k}{n}\right) := -\frac{1}{(N-1)!} \int_x^{\frac{k}{n}} \left(e^{-\frac{k}{n}} - e^{-s}\right)^{N-1} \left[F^{(N)}(e^{-s}) - F^{(N)}(e^{-x})\right] e^{-s} ds.$$

Thus

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) f\left(\frac{k}{n}\right) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) = \\ \sum_{j=1}^N \frac{F^{(j)}(e^{-x})}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \left(e^{-\frac{k}{n}} - e^{-x}\right)^j + \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \frac{1}{(N-1)!} \int_x^{\frac{k}{n}} \left(e^{-\frac{k}{n}} - e^{-s}\right)^{N-1} \left[F^{(N)}(e^{-s}) - F^{(N)}(e^{-x})\right] de^{-s}.$$

Call

$$S_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) W(nx - k). \quad (58)$$

Therefore

$$S_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \right) = \sum_{j=1}^N \frac{F^{(j)}(e^{-x})}{j!} S_n^* \left((e^{-\cdot} - e^{-x})^j, x \right) + U_n^*(x),$$

where

$$U_n^*(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \mu^*,$$

with

$$\mu^* := \frac{1}{(N-1)!} \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w \right)^{N-1} \left[F^{(N)}(w) - F^{(N)}(e^{-x}) \right] dw.$$

Case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$.

i) Subcase of $x \geq \frac{k}{n}$. I.e. $e^{-\frac{k}{n}} \geq e^{-x}$.

$$\begin{aligned} |\mu^*| &\leq \frac{1}{(N-1)!} \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w \right)^{N-1} \left| F^{(N)}(w) - F^{(N)}(e^{-x}) \right| dw \leq \\ &\frac{1}{(N-1)!} \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w \right)^{N-1} \omega_1 \left(F^{(N)}, |w - e^{-x}| \right) dw \leq \\ &\frac{1}{(N-1)!} \omega_1 \left(F^{(N)}, \left| e^{-\frac{k}{n}} - e^{-x} \right| \right) \int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w \right)^{N-1} dw \leq \\ &\frac{\left(e^{-\frac{k}{n}} - e^{-x} \right)^N}{N!} \omega_1 \left(F^{(N)}, e^{-a} \left| x - \frac{k}{n} \right| \right) \leq \\ &e^{-aN} \frac{\left| x - \frac{k}{n} \right|^N}{N!} \omega_1 \left(F^{(N)}, e^{-a} \left| x - \frac{k}{n} \right| \right) \leq \\ &\frac{e^{-aN}}{N! n^{\alpha N}} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right). \end{aligned}$$

Hence when $x \geq \frac{k}{n}$ we get

$$|\mu^*| \leq \frac{e^{-aN}}{N! n^{\alpha N}} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right). \quad (59)$$

ii) Subcase of $\frac{k}{n} \geq x$. Then $e^{-\frac{k}{n}} \leq e^{-x}$ and

$$\begin{aligned} |\mu^*| &\leq \frac{1}{(N-1)!} \int_{e^{-\frac{k}{n}}}^{e^{-x}} \left(w - e^{-\frac{k}{n}} \right)^{N-1} \left| F^{(N)}(w) - F^{(N)}(e^{-x}) \right| dw \leq \\ &\frac{1}{(N-1)!} \int_{e^{-\frac{k}{n}}}^{e^{-x}} \left(w - e^{-\frac{k}{n}} \right)^{N-1} \omega_1 \left(F^{(N)}, |w - e^{-x}| \right) dw \leq \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{(N-1)!} \omega_1 \left(F^{(N)}, e^{-x} - e^{-\frac{k}{n}} \right) \int_{e^{-\frac{k}{n}}}^{e^{-x}} \left(w - e^{-\frac{k}{n}} \right)^{N-1} dw \leq \\
 & \frac{1}{(N-1)!} \omega_1 \left(F^{(N)}, e^{-a} \left| x - \frac{k}{n} \right| \right) \frac{\left(e^{-x} - e^{-\frac{k}{n}} \right)^N}{N} \leq \\
 & \frac{1}{N!} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right) e^{-aN} \left| x - \frac{k}{n} \right|^N \leq \\
 & \frac{1}{N!} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right) \frac{e^{-aN}}{n^{\alpha N}},
 \end{aligned}$$

i.e.

$$|\mu^*| \leq \frac{e^{-aN}}{N! n^{\alpha N}} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right), \quad (60)$$

when $\frac{k}{n} \geq x$. So in general when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$ we proved that

$$|\mu^*| \leq \frac{e^{-aN}}{N! n^{\alpha N}} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right). \quad (61)$$

Also we observe:

i)' When $\frac{k}{n} \leq x$, we get

$$\begin{aligned}
 |\mu^*| & \leq \frac{1}{(N-1)!} \left(\int_{e^{-x}}^{e^{-\frac{k}{n}}} \left(e^{-\frac{k}{n}} - w \right)^{N-1} dw \right) 2 \|F^{(N)}\|_\infty \\
 & = \frac{2 \|F^{(N)}\|_\infty}{(N-1)!} \frac{\left(e^{-\frac{k}{n}} - e^{-x} \right)^N}{N} \leq \frac{2 \|F^{(N)}\|_\infty}{N!} \left(e^{-a} - e^{-b} \right)^N.
 \end{aligned}$$

ii)' When $\frac{k}{n} \geq x$, we obtain

$$\begin{aligned}
 |\mu^*| & \leq \frac{1}{(N-1)!} \left(\int_{e^{-\frac{k}{n}}}^{e^{-x}} \left(w - e^{-\frac{k}{n}} \right)^{N-1} dw \right) 2 \|F^{(N)}\|_\infty \\
 & = \frac{2 \|F^{(N)}\|_\infty}{N!} \left(e^{-x} - e^{-\frac{k}{n}} \right)^N \leq \frac{2 \|F^{(N)}\|_\infty}{N!} \left(e^{-a} - e^{-b} \right)^N.
 \end{aligned}$$

We proved always true that

$$|\mu^*| \leq \frac{2 \left(e^{-a} - e^{-b} \right)^N \|F^{(N)}\|_\infty}{N!}. \quad (62)$$

Consequently, we get

$$\begin{aligned}
 |U_n^*(x)| & \leq \sum_{\substack{k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} W(nx - k) |\mu^*| + \sum_{\substack{k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} W(nx - k) |\mu^*| \leq \\
 & \left(\frac{e^{-aN}}{N! n^{\alpha N}} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right) \right) \left(\sum_{k=-\infty}^{\infty} W(nx - k) \right) +
 \end{aligned}$$

$$\left(\frac{2(e^{-a} - e^{-b})^N \|F^{(N)}\|_\infty}{N!} \right) \left(\sum_{\substack{k=-\infty \\ |k-nx| > n^{1-\alpha}}}^{\infty} W(nx-k) \right) \stackrel{(\text{by (11), (17)})}{\leq} \quad (63)$$

$$\frac{e^{-aN}}{N!n^{\alpha N}} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right) + \left(\frac{2(e^{-a} - e^{-b})^N \|F^{(N)}\|_\infty}{N!} \right) \gamma A^{-\lambda(n^{1-\alpha}-2)}.$$

So we have proved that

$$|U_n^*(x)| \leq \frac{e^{-aN}}{N!n^{\alpha N}} \omega_1 \left(F^{(N)}, \frac{e^{-a}}{n^\alpha} \right) + \frac{2(e^{-a} - e^{-b})^N \|F^{(N)}\|_\infty}{N!} \gamma A^{-\lambda(n^{1-\alpha}-2)}.$$

We also notice that

$$\begin{aligned} & \left| S_n^* \left((e^{-\cdot} - e^{-x})^j, x \right) \right| \leq S_n^* \left(|e^{-\cdot} - e^{-x}|^j, x \right) \leq \\ & e^{-aj} S_n^* \left(|\cdot - x|^j, x \right) = e^{-aj} \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \left| \frac{k}{n} - x \right|^j \right) = e^{-aj}. \end{aligned}$$

$$\left[\sum_{\substack{k=\lceil na \rceil \\ |x - \frac{k}{n}| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} W(nx-k) \left| \frac{k}{n} - x \right|^j + \sum_{\substack{k=\lceil na \rceil \\ |x - \frac{k}{n}| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} W(nx-k) \left| \frac{k}{n} - x \right|^j \right] \stackrel{(11)}{\leq} \quad (64)$$

$$e^{-aj} \left[\frac{1}{n^{\alpha j}} + (b-a)^j \sum_{\substack{k=\lceil na \rceil \\ |x - \frac{k}{n}| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} W(nx-k) \right] \stackrel{(17)}{\leq}$$

$$e^{-aj} \left[\frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right].$$

Thus we have proved

$$\left| S_n^* \left((e^{-\cdot} - e^{-x})^j, x \right) \right| \leq e^{-aj} \left[\frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right],$$

for $j = 1, \dots, N$, and the theorem. $\square$

We continue with

Theorem 3.5. *Let $f \in C^N \left(\left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon \right] \right)$, $n, N \in \mathbb{N}$, $0 < \varepsilon < \frac{\pi}{2}$, ε small, $x \in \left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon \right]$, $0 < \alpha < 1$. Then*

1)

$$\left| S_n(f, x) - \sum_{j=1}^N \frac{(f \circ \sin^{-1})^{(j)}(\sin x)}{j!} S_n((\sin \cdot - \sin x)^j, x) - f(x) \right| \leq \quad (65)$$

$$K(q) \left[\frac{\omega_1\left((f \circ \sin^{-1})^{(N)}, \frac{1}{n^\alpha}\right)}{n^{\alpha N} N!} + \left(\frac{2^{N+1} \left\| (f \circ \sin^{-1})^{(N)} \right\|_\infty}{N!} \right) \gamma A^{-\lambda(n^{1-\alpha}-2)} \right],$$

2)

$$|S_n(f, x) - f(x)| \leq K(q). \quad (66)$$

$$\left\{ \sum_{j=1}^N \frac{|(f \circ \sin^{-1})^{(j)}(\sin x)|}{j!} \left[\frac{1}{n^{\alpha j}} + (\pi - 2\varepsilon)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] + \left[\frac{\omega_1\left((f \circ \sin^{-1})^{(N)}, \frac{1}{n^\alpha}\right)}{N! n^{\alpha N}} + \left(\frac{2^{N+1} \left\| (f \circ \sin^{-1})^{(N)} \right\|_\infty}{N!} \right) \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] \right\},$$

3) assume further $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$ for some $x_0 \in [-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon]$, it holds

$$|S_n(f, x_0) - f(x_0)| \leq K(q). \quad (67)$$

$$\left[\frac{\omega_1\left((f \circ \sin^{-1})^{(N)}, \frac{1}{n^\alpha}\right)}{n^{\alpha N} N!} + \left(\frac{2^{N+1} \left\| (f \circ \sin^{-1})^{(N)} \right\|_\infty}{N!} \right) \gamma A^{-\lambda(n^{1-\alpha}-2)} \right].$$

Notice in the last the high speed of convergence of order $n^{-\alpha(N+1)}$.

Proof. As it is long and similar to the proof of Theorem 3.4 it is omitted. $\square$

We present also

Theorem 3.6. Let $f \in C^N([\varepsilon, \pi - \varepsilon])$, $n, N \in \mathbb{N}$, $\varepsilon > 0$ small, $x \in [\varepsilon, \pi - \varepsilon]$, $0 < \alpha < 1$. Then

1)

$$\left| S_n(f, x) - \sum_{j=1}^N \frac{(f \circ \cos^{-1})^{(j)}(\cos x)}{j!} S_n((\cos \cdot - \cos x)^j, x) - f(x) \right| \leq \quad (68)$$

$$K(q) \left[\frac{\omega_1\left((f \circ \cos^{-1})^{(N)}, \frac{1}{n^\alpha}\right)}{n^{\alpha N} N!} + \left(\frac{2^{N+1} \left\| (f \circ \cos^{-1})^{(N)} \right\|_\infty}{N!} \right) \gamma A^{-\lambda(n^{1-\alpha}-2)} \right],$$

2)

$$|S_n(f, x) - f(x)| \leq K(q). \quad (69)$$

$$\left\{ \sum_{j=1}^N \frac{|(f \circ \cos^{-1})^{(j)}(\cos x)|}{j!} \left[\frac{1}{n^{\alpha j}} + (\pi - 2\varepsilon)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] + \left[\frac{\omega_1 \left((f \circ \cos^{-1})^{(N)}, \frac{1}{n^\alpha} \right)}{N! n^{\alpha N}} + \left(\frac{2^{N+1}}{N!} \left\| (f \circ \cos^{-1})^{(N)} \right\|_\infty \right) \gamma A^{-\lambda(n^{1-\alpha}-2)} \right] \right\},$$

3) assume further $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$ for some $x_0 \in [\varepsilon, \pi - \varepsilon]$, it holds

$$|S_n(f, x_0) - f(x_0)| \leq K(q). \quad (70)$$

$$\left[\frac{\omega_1 \left((f \circ \cos^{-1})^{(N)}, \frac{1}{n^\alpha} \right)}{n^{\alpha N} N!} + \left(\frac{2^{N+1}}{N!} \left\| (f \circ \cos^{-1})^{(N)} \right\|_\infty \right) \gamma A^{-\lambda(n^{1-\alpha}-2)} \right].$$

Notice in the last the high speed of convergence of order $n^{-\alpha(N+1)}$.

Proof. As it is long and similar to the proof of Theorem 3.4 it is omitted. $\square$

4. ABOUT GENERALIZED FRACTIONAL CALCULUS

We will use

Definition 4.1. ([9], p. 106) Let $\beta > 0$, $[\beta] = N$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^N([a, b], X)$, where $[a, b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$.

We define the left generalized g -fractional derivative X -valued of f of order β as follows:

$$\left(D_{*a;g}^\beta f \right)(x) := \frac{1}{\Gamma(N - \beta)} \int_a^x (g(x) - g(t))^{N-\beta-1} g'(t) (f \circ g^{-1})^{(N)}(g(t)) dt, \quad (71)$$

$\forall x \in [a, b]$. The last integral is of Bochner type.

If $\beta \notin \mathbb{N}$, by [9], p. 98, we have that $\left(D_{*a;g}^\beta f \right) \in C([a, b], X)$.

We set

$$D_{*a;g}^N f(x) := \left((f \circ g^{-1})^{(N)} \circ g \right)(x) \in C([a, b], X), \quad N \in \mathbb{N}, \quad (72)$$

$$D_{*a;g}^0 f(x) = f(x), \quad \forall x \in [a, b].$$

When $g = id$, then

$$D_{*a;g}^\beta f = D_{*a;id}^\beta f = D_{*a}^\beta f, \quad (73)$$

the usual left X -valued Caputo fractional derivative, see [9], p. 128.

We will also use

Definition 4.2. ([9], p. 107) Let $\beta > 0$, $[\beta] = N$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^N([a, b], X)$, where $[a, b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$.

We define the right generalized g -fractional derivative X -valued of f of order β as follows:

$$\left(D_{b-;g}^\beta f\right)(x) := \frac{(-1)^N}{\Gamma(N-\beta)} \int_x^b (g(t) - g(x))^{N-\beta-1} g'(t) (f \circ g^{-1})^{(N)}(g(t)) dt, \quad (74)$$

$\forall x \in [a, b]$. The last integral is of Bochner type.

If $\beta \notin \mathbb{N}$, by [9], p. 98, we have that $\left(D_{b-;g}^\beta f\right) \in C([a, b], X)$.

We set

$$D_{b-;g}^N f(x) := (-1)^N \left((f \circ g^{-1})^{(N)} \circ g\right)(x) \in C([a, b], X), \quad N \in \mathbb{N}, \quad (75)$$

$$D_{b-;g}^0 f(x) = f(x), \quad \forall x \in [a, b].$$

When $g = id$, then

$$D_{b-;g}^\beta f = D_{b-;id}^\beta f = D_{b-}^\beta f, \quad (76)$$

the usual left X -valued Caputo fractional derivative, see [8].

We mention the following g -left generalized X -valued Taylor's formula:

Theorem 4.1. ([9], p. 107) Let $\beta > 0$, $N = [\beta]$ and $f \in C^N([a, b], X)$, where $[a, b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$. Then

$$\begin{aligned} f(x) &= f(a) + \sum_{j=1}^{N-1} \frac{(g(x) - g(a))^j}{j!} (f \circ g^{-1})^{(j)}(g(a)) + \\ &\quad \frac{1}{\Gamma(\beta)} \int_a^x (g(x) - g(t))^{\beta-1} g'(t) \left(D_{a+;g}^\beta f\right)(t) dt = \\ &\quad f(a) + \sum_{j=1}^{N-1} \frac{(g(x) - g(a))^j}{j!} (f \circ g^{-1})^{(j)}(g(a)) + \\ &\quad \frac{1}{\Gamma(\beta)} \int_{g(a)}^{g(x)} (g(x) - z)^{\beta-1} \left(\left(D_{a+;g}^\beta f\right) \circ g^{-1}\right)(z) dz, \quad \forall x \in [a, b]. \end{aligned} \quad (77)$$

We also mention the following g -right generalized X -valued Taylor's formula:

Theorem 4.2. ([9], p. 108) Let $\beta > 0$, $N = [\beta]$ and $f \in C^N([a, b], X)$, where $[a, b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$. Then

$$\begin{aligned} f(x) &= f(b) + \sum_{j=1}^{N-1} \frac{(g(x) - g(b))^j}{j!} (f \circ g^{-1})^{(j)}(g(b)) + \\ &\quad \frac{1}{\Gamma(\beta)} \int_x^b (g(t) - g(x))^{\beta-1} g'(t) \left(D_{b-;g}^\beta f\right)(t) dt = \end{aligned}$$

$$f(b) + \sum_{j=1}^{N-1} \frac{(g(x) - g(b))^j}{j!} (f \circ g^{-1})^{(j)}(g(b)) + \frac{1}{\Gamma(\beta)} \int_{g(x)}^{g(b)} (z - g(x))^{\beta-1} \left((D_{b-;g}^\beta f) \circ g^{-1} \right)(z) dz, \quad \forall x \in [a, b]. \quad (78)$$

If $f^{(j)}(x_0) = 0$, $j = 1, \dots, N-1$, then by [16], p. 84, vectorial product rule and chain rule, we derive that $(f \circ g^{-1})^{(j)}(g(x_0)) = 0$, $j = 1, \dots, N-1$, $x_0 \in [a, b]$. See also [6], p. 401.

Integrals in (71)-(78) are of Bochner type. For the Bochner integral excellent resources are [15] and [1, pp. 422-428].

When $X = \mathbb{R}$, then the Bochner integral collapses to the Lebesgue integral. So all the above apply also to real valued functions which we treat here.

Example 4.3. The following function

$$f(x) = x + \sqrt{x^2 + 1}, \quad x \in \mathbb{R},$$

has derivative $f'(x) = 1 + \frac{x}{\sqrt{x^2+1}}$ with range $(0, 2)$.

Indeed we have: $1 + \frac{x}{\sqrt{x^2+1}} > 0$, iff $\frac{x}{\sqrt{x^2+1}} > -1$, iff $x > -\sqrt{x^2+1}$, iff $-x < \sqrt{x^2+1}$. The last is true if $x \geq 0$, and if $x < 0$, then $|x| < \sqrt{x^2+1}$, iff $x^2 < x^2+1$, iff $0 < 1$ true.

And we have $1 + \frac{x}{\sqrt{x^2+1}} < 2$, iff $\frac{x}{\sqrt{x^2+1}} < 1$, iff $x < \sqrt{x^2+1}$, if $x < 0$ the last is true, if $x > 0$ then $x^2 < x^2+1$, iff $0 < 1$ true. Thus, f is strictly increasing, differentiable and uniformly continuous. It has C^1 inverse, i.e. f^{-1} is continuous and differentiable, with $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$, by continuous f' and $f' \neq 0$.

Example 4.4. Let $f(x) = x + \arctan(x)$. Then $f'(x) = 1 + \frac{1}{1+x^2}$, with $1 < f'(x) \leq 2$, $\forall x \in \mathbb{R}$. Thus f is C^1 , strictly increasing and uniformly continuous.

By the inverse function theorem f^{-1} is differentiable and $(f^{-1})'$ is continuous.

Example 4.5. Let $f(x) = 2x + \sin(x)$, it is $1 \leq f'(x) \leq 3$, thus f is uniformly continuous, strictly increasing, continuously differentiable with a C^1 inverse.

So our assumption on g in Definitions 4.1, 4.2 and Theorems 4.1, 4.2 is possible.

5. GENERALIZED FRACTIONAL SYMMETRIZED NEURAL NETWORK APPROXIMATION

We present the following:

Theorem 5.1. Let $0 < \alpha < 1$, $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$ strictly increasing, such that $g^{-1} \in C^N([g(a), g(b)])$. Let also $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then

1)

$$\left\| S_n(f, x) - \sum_{j=1}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} S_n\left((g(\cdot) - g(x))^j\right)(x) - f(x) \right\| \leq$$

$$\frac{K(q)}{\Gamma(\beta+1)} \left\{ \left(\omega_1 \left(D_{x-;gf}^\beta; \frac{1}{n^\alpha} \right)_{[a,x]} + \omega_1 \left(D_{*x;gf}^\beta; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \right. \\ \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\|D_{x-;gf}^\beta\|_{\infty,[a,x]} (g(x) - g(a))^\beta + \|D_{*x;gf}^\beta\|_{\infty,[x,b]} (g(b) - g(x))^\beta \right) \right\}, \quad (79)$$

2) if $f^{(j)}(x) = 0$, $j = 1, \dots, N-1$, we have that

$$\|S_n(f, x) - f(x)\| \leq \\ \frac{K(q)}{\Gamma(\beta+1)} \inf_g \left\{ \left(\omega_1 \left(D_{x-;gf}^\beta; \frac{1}{n^\alpha} \right)_{[a,x]} + \omega_1 \left(D_{*x;gf}^\beta; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \right. \\ \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\|D_{x-;gf}^\beta\|_{\infty,[a,x]} (g(x) - g(a))^\beta + \|D_{*x;gf}^\beta\|_{\infty,[x,b]} (g(b) - g(x))^\beta \right) \right\}, \quad (80)$$

the speed of convergence to zero here is $\frac{1}{n^{\alpha(\beta+1)}}$,

$$3) \\ \|S_n(f, x) - f(x)\| \leq \\ K(q) \inf_g \left\{ \sum_{j=1}^{N-1} \frac{\|(f \circ g^{-1})^{(j)}(g(x))\|}{j!} \|g^{(1)}\|_\infty^j \left\{ \frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right\} \right. \\ \left. + \frac{1}{\Gamma(\beta+1)} \left\{ \left(\omega_1 \left(D_{x-;gf}^\beta; \frac{1}{n^\alpha} \right)_{[a,x]} + \omega_1 \left(D_{*x;gf}^\beta; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \right. \right. \\ \left. \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\|D_{x-;gf}^\beta\|_{\infty,[a,x]} (g(x) - g(a))^\beta + \|D_{*x;gf}^\beta\|_{\infty,[x,b]} (g(b) - g(x))^\beta \right) \right\} \right\}, \quad (81)$$

$\forall x \in [a, b]$,

$$4) \\ \|S_n(f) - f\|_\infty \leq \\ K(q) \inf_g \left\{ \sum_{j=1}^{N-1} \frac{\|(f \circ g^{-1})^{(j)} \circ g\|_\infty}{j!} \|g^{(1)}\|_\infty^j \left\{ \frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right\} \right. \\ \left. + \frac{1}{\Gamma(\beta+1)} \left\{ \left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-;gf}^\beta; \frac{1}{n^\alpha} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x;gf}^\beta; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \right. \right. \\ \left. \left. (g(b) - g(a))^\beta \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\sup_{x \in [a,b]} \|D_{x-;gf}^\beta\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \|D_{*x;gf}^\beta\|_{\infty,[x,b]} \right) \right\} \right\}. \quad (82)$$

Above, when $N = 1$, the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain X -valued generalize fractional type pointwise and uniform convergence with rates $S_n \rightarrow I$, the unit operator, as $n \rightarrow \infty$.

Proof. Let $x \in [a, b]$. We have that $D_{x-;g}^\beta f(x) = D_{*x;g}^\beta f(x) = 0$, explanation at the end of this proof.

From Theorem 4.1, we get by the g -left generalized X -valued Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} \left(g\left(\frac{k}{n}\right) - g(x)\right)^j + \frac{1}{\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(g\left(\frac{k}{n}\right) - g(J)\right)^{\beta-1} g'(J) \left(D_{*x;g}^\beta f(J) - D_{*x;g}^\beta f(x)\right) dJ, \quad (83)$$

for all $x \leq \frac{k}{n} \leq b$.

Also from Theorem 4.2, using the g -right generalized X -valued Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} \left(g\left(\frac{k}{n}\right) - g(x)\right)^j + \frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right)\right)^{\beta-1} g'(J) \left(D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x)\right) dJ, \quad (84)$$

for all $a \leq \frac{k}{n} \leq x$.

Hence we have

$$f\left(\frac{k}{n}\right) W(nx - k) = \sum_{j=0}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} W(nx - k) \left(g\left(\frac{k}{n}\right) - g(x)\right)^j + \frac{W(nx - k)}{\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(g\left(\frac{k}{n}\right) - g(J)\right)^{\beta-1} g'(J) \left(D_{*x;g}^\beta f(J) - D_{*x;g}^\beta f(x)\right) dJ, \quad (85)$$

for all $x \leq \frac{k}{n} \leq b$, iff $\lceil nx \rceil \leq k \leq \lfloor nb \rfloor$,

and

$$f\left(\frac{k}{n}\right) W(nx - k) = \sum_{j=0}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} W(nx - k) \left(g\left(\frac{k}{n}\right) - g(x)\right)^j + \frac{W(nx - k)}{\Gamma(\beta)} \int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right)\right)^{\beta-1} g'(J) \left(D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x)\right) dJ, \quad (86)$$

for all $a \leq \frac{k}{n} \leq x$, iff $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$.

Therefore it holds

$$\sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) W(nx - k) = \sum_{j=0}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} \left(\sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} W(nx - k) \left(g\left(\frac{k}{n}\right) - g(x)\right)^j \right)$$

$$+\frac{1}{\Gamma(\beta)} \sum_{k=\lfloor nx \rfloor +1}^{\lfloor nb \rfloor} W(nx-k) \quad (87)$$

$$\int_x^{\frac{k}{n}} \left(g\left(\frac{k}{n}\right) - g(J) \right)^{\beta-1} g'(J) \left(D_{*x;g}^\beta f(J) - D_{*x;g}^\beta f(x) \right) dJ,$$

and

$$\begin{aligned} & \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} f\left(\frac{k}{n}\right) W(nx-k) = \\ & \sum_{j=0}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} \left(\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} W(nx-k) \left(g\left(\frac{k}{n}\right) - g(x) \right)^j \right) \\ & + \frac{1}{\Gamma(\beta)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} W(nx-k) \quad (88) \\ & \int_x^{\frac{k}{n}} \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \left(D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x) \right) dJ. \end{aligned}$$

Adding the last two equations (87) and (88) we obtain

$$\begin{aligned} S_n^*(f, x) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) W(nx-k) = \\ & \sum_{j=0}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \left(g\left(\frac{k}{n}\right) - g(x) \right)^j \\ & + \frac{1}{\Gamma(\beta)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} W(nx-k) \right. \\ & \left. \int_x^{\frac{k}{n}} \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \left(D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x) \right) dJ \right. \\ & \left. + \sum_{k=\lfloor nx \rfloor +1}^{\lfloor nb \rfloor} W(nx-k) \right. \\ & \left. \int_x^{\frac{k}{n}} \left(g\left(\frac{k}{n}\right) - g(J) \right)^{\beta-1} g'(J) \left(D_{*x;g}^\beta f(J) - D_{*x;g}^\beta f(x) \right) dJ \right\}. \quad (89) \end{aligned}$$

So we have derived

$$\begin{aligned} & S_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \right) - \\ & \sum_{j=1}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} S_n^* \left((g(\cdot) - g(x))^j \right) = u_n^*(x), \quad (90) \end{aligned}$$

where

$$u_n^*(x) := \frac{1}{\Gamma(\beta)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} W(nx - k) \int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \left(D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x) \right) dJ + \sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} W(nx - k) \int_x^{\frac{k}{n}} \left(g\left(\frac{k}{n}\right) - g(J) \right)^{\beta-1} g'(J) \left(D_{*x;g}^\beta f(J) - D_{*x;g}^\beta f(x) \right) dJ \right\}. \quad (91)$$

We set

$$u_{1n}^*(x) := \frac{1}{\Gamma(\beta)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} W(nx - k) \int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \left(D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x) \right) dJ, \quad (92)$$

and

$$u_{2n}^*(x) := \frac{1}{\Gamma(\beta)} \sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} W(nx - k) \int_x^{\frac{k}{n}} \left(g\left(\frac{k}{n}\right) - g(J) \right)^{\beta-1} g'(J) \left[D_{*x;g}^\beta f(J) - D_{*x;g}^\beta f(x) \right] dJ. \quad (93)$$

That is

$$u_n^*(x) = u_{1n}^*(x) + u_{2n}^*(x). \quad (94)$$

We assume $b - a > \frac{1}{n^\alpha}$, $0 < \alpha < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b - a)^{-\frac{1}{\alpha}} \right\rceil$. It is always true that either $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}$.

For $k = \lceil na \rceil, \dots, \lfloor nx \rfloor$, we consider

$$\begin{aligned} \gamma_{1k}^* &:= \left\| \int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \left(D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x) \right) dJ \right\| = \\ &\left\| \int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) D_{x-;g}^\beta f(J) dJ \right\| \leq \\ &\int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \left\| D_{x-;g}^\beta f(J) \right\| dJ \leq \\ &\left\| D_{x-;g}^\beta f(J) \right\|_{\infty, [a, x]} \int_{g(\frac{k}{n})}^{g(x)} \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} dg(J) = \\ &\left\| D_{x-;g}^\beta f(J) \right\|_{\infty, [a, x]} \frac{(g(x) - g(\frac{k}{n}))^\beta}{\beta} \leq \end{aligned} \quad (95)$$

$$\left\| D_{x-;g}^\beta f(J) \right\|_{\infty, [a,x]} \frac{(g(x) - g(a))^\beta}{\beta}.$$

That is

$$\gamma_{1k}^* \leq \left\| D_{x-;g}^\beta f(J) \right\|_{\infty, [a,x]} \frac{(g(x) - g(a))^\beta}{\beta}, \quad (96)$$

for $k = \lceil na \rceil, \dots, \lfloor nx \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$ that

$$\begin{aligned} \gamma_{1k}^* &\leq \int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \left\| D_{x-;g}^\beta f(J) - D_{x-;g}^\beta f(x) \right\| dJ \leq \\ &\int_{\frac{k}{n}}^x \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} g'(J) \omega_1 \left(D_{x-;g}^\beta f, |J - x| \right)_{[a,x]} dJ \leq \\ &\omega_1 \left(D_{x-;g}^\beta f, \left| x - \frac{k}{n} \right| \right)_{[a,x]} \int_{g(\frac{k}{n})}^{g(x)} \left(g(J) - g\left(\frac{k}{n}\right) \right)^{\beta-1} dg(J) \leq \\ &\omega_1 \left(D_{x-;g}^\beta f, \frac{1}{n^\alpha} \right)_{[a,x]} \frac{(g(x) - g(\frac{k}{n}))^\beta}{\beta} \leq \\ &\omega_1 \left(D_{x-;g}^\beta f, \frac{1}{n^\alpha} \right)_{[a,x]} \frac{\|g^{(1)}\|_\infty^\beta \left| x - \frac{k}{n} \right|^\beta}{\beta} \leq \\ &\omega_1 \left(D_{x-;g}^\beta f, \frac{1}{n^\alpha} \right)_{[a,x]} \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}\beta}. \end{aligned} \quad (97)$$

We have proved that

$$\gamma_{1k}^* \leq \omega_1 \left(D_{x-;g}^\beta f, \frac{1}{n^\alpha} \right)_{[a,x]} \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}\beta}, \quad (98)$$

when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$.

Consequently we obtain

$$\begin{aligned} \|u_{1n}^*(x)\| &\leq \frac{1}{\Gamma(\beta)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} W(nx - k) \gamma_{1k}^* = \\ &\frac{1}{\Gamma(\beta)} \left\{ \sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nx \rfloor} W(nx - k) \gamma_{1k}^* + \sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\lfloor nx \rfloor} W(nx - k) \gamma_{1k}^* \right\} \leq \end{aligned}$$

$$\begin{aligned}
& \frac{1}{\Gamma(\beta)} \left\{ \left(\sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ |\frac{k}{n} - x| \leq \frac{1}{n^\alpha} \end{array} \right\}}^{\lfloor nx \rfloor} W(nx - k) \right) \omega_1 \left(D_{x-;gf}^\beta, \frac{1}{n^\alpha} \right)_{[a,x]} \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}\beta} \right. \\
& + \left. \left(\sum_{\left\{ \begin{array}{l} k = \lceil na \rceil \\ |\frac{k}{n} - x| > \frac{1}{n^\alpha} \end{array} \right\}}^{\lfloor nx \rfloor} W(nx - k) \right) \left\| D_{x-;gf}^\beta(J) \right\|_{\infty,[a,x]} \frac{(g(x) - g(a))^\beta}{\beta} \right\} \stackrel{\text{(by (11), (17))}}{\leq} \\
& \frac{1}{\Gamma(\beta+1)} \left\{ \omega_1 \left(D_{x-;gf}^\beta, \frac{1}{n^\alpha} \right)_{[a,x]} \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \gamma A^{-\lambda(n^{1-\alpha}-2)} \left\| D_{x-;gf}^\beta \right\|_{\infty,[a,x]} (g(x) - g(a))^\beta \right\}.
\end{aligned} \tag{99}$$

Therefore it holds

$$\begin{aligned}
& \|u_{1n}^*(x)\| \leq \frac{1}{\Gamma(\beta+1)} \\
& \left\{ \omega_1 \left(D_{x-;gf}^\beta, \frac{1}{n^\alpha} \right)_{[a,x]} \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \gamma A^{-\lambda(n^{1-\alpha}-2)} \left\| D_{x-;gf}^\beta \right\|_{\infty,[a,x]} (g(x) - g(a))^\beta \right\}.
\end{aligned} \tag{100}$$

For $k = \lfloor nx \rfloor + 1, \dots, \lfloor nb \rfloor$, we consider

$$\gamma_{2k}^* := \left\| \int_x^{\frac{k}{n}} \left(g\left(\frac{k}{n}\right) - g(J) \right)^{\beta-1} g'(J) \left(D_{*x;gf}^\beta(J) - D_{*x;gf}^\beta(x) \right) dJ \right\|, \tag{101}$$

as earlier we find that

$$\gamma_{2k}^* \leq \left\| D_{*x;gf}^\beta \right\|_{\infty,[x,b]} \frac{(g(b) - g(x))^\beta}{\beta}. \tag{102}$$

In case of $|\frac{k}{n} - x| \leq \frac{1}{n^\alpha}$, we find as earlier

$$\gamma_{2k}^* \leq \omega_1 \left(D_{*x;gf}^\beta, \frac{1}{n^\alpha} \right)_{[x,b]} \frac{\|g^{(1)}\|_\infty^\beta}{\beta n^{\alpha\beta}}. \tag{103}$$

And, as earlier, we get that

$$\begin{aligned}
& \|u_{2n}^*(x)\| \leq \frac{1}{\Gamma(\beta+1)} \\
& \left\{ \omega_1 \left(D_{*x;gf}^\beta, \frac{1}{n^\alpha} \right)_{[x,b]} \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \gamma A^{-\lambda(n^{1-\alpha}-2)} \left\| D_{*x;gf}^\beta \right\|_{\infty,[x,b]} (g(b) - g(x))^\beta \right\}.
\end{aligned} \tag{104}$$

Therefore

$$\|u_n^*(x)\| \leq \|u_{1n}^*(x)\| + \|u_{2n}^*(x)\| \leq$$

$$\frac{1}{\Gamma(\beta+1)} \left\{ \left(\omega_1 \left(D_{x-;g}^\beta f, \frac{1}{n^\alpha} \right)_{[a,x]} + \omega_1 \left(D_{*x;g}^\beta f, \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \right. \\ \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\|D_{x-;g}^\beta\|_{\infty,[a,x]} (g(x) - g(a))^\beta + \|D_{*x;g}^\beta\|_{\infty,[x,b]} (g(b) - g(x))^\beta \right) \right\}. \quad (105)$$

Next we estimate

$$\begin{aligned} & \left| S_n^* \left((g(\cdot) - g(x))^j \right) \right| = \\ & \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \left(g\left(\frac{k}{n}\right) - g(x) \right)^j \right| \leq \\ & \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \left| g\left(\frac{k}{n}\right) - g(x) \right|^j \leq \\ & \|g^{(1)}\|_\infty^j \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \left| \frac{k}{n} - x \right|^j = \\ & \|g^{(1)}\|_\infty^j \left\{ \sum_{\substack{k=\lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} W(nx - k) \left| \frac{k}{n} - x \right|^j + \right. \\ & \left. \sum_{\substack{k=\lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} W(nx - k) \left| \frac{k}{n} - x \right|^j \right\} \stackrel{(17)}{\leq} \\ & \|g^{(1)}\|_\infty^j \left\{ \frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right\}. \end{aligned} \quad (106)$$

That is

$$\begin{aligned} & \left| S_n^* \left((g(\cdot) - g(x))^j \right) \right| \leq \\ & \|g^{(1)}\|_\infty^j \left\{ \frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right\}, \end{aligned} \quad (107)$$

for $j = 1, \dots, N-1, \forall x \in [a, b]$.

Putting things together, we have established

$$\left\| S_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \right) \right\| \leq$$

$$\begin{aligned}
& \sum_{j=1}^{N-1} \frac{\left\| (f \circ g^{-1})^{(j)}(g(x)) \right\|}{j!} \left\| g^{(1)} \right\|_{\infty}^j \left\{ \frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right\} \\
& + \frac{1}{\Gamma(\beta+1)} \left\{ \left(\omega_1 \left(D_{x-;g}^{\beta} f; \frac{1}{n^{\alpha}} \right)_{[a,x]} + \omega_1 \left(D_{*x;g}^{\beta} f; \frac{1}{n^{\alpha}} \right)_{[x,b]} \right) \frac{\left\| g^{(1)} \right\|_{\infty}^{\beta}}{n^{\alpha\beta}} + \right. \\
& \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\left\| D_{x-;g}^{\beta} f \right\|_{\infty,[a,x]} (g(x) - g(a))^{\beta} + \left\| D_{*x;g}^{\beta} f \right\|_{\infty,[x,b]} (g(b) - g(x))^{\beta} \right) \right\} \\
& =: K_{gn}^{*}(x). \tag{108}
\end{aligned}$$

As a result we derive, by (25),

$$\|S_n(f, x) - f(x)\| \leq K(q) K_{gn}^{*}(x), \tag{109}$$

$\forall x \in [a, b]$.

Furthermore we have that

$$\begin{aligned}
\|K_{gn}^{*}\|_{\infty} & \leq \sum_{j=1}^{N-1} \frac{\left\| (f \circ g^{-1})^{(j)} \circ g \right\|_{\infty}}{j!} \left\| g^{(1)} \right\|_{\infty}^j \left\{ \frac{1}{n^{\alpha j}} + (b-a)^j \gamma A^{-\lambda(n^{1-\alpha}-2)} \right\} \\
& + \frac{1}{\Gamma(\beta+1)} \left\{ \left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-;g}^{\beta} f; \frac{1}{n^{\alpha}} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x;g}^{\beta} f; \frac{1}{n^{\alpha}} \right)_{[x,b]} \right) \frac{\left\| g^{(1)} \right\|_{\infty}^{\beta}}{n^{\alpha\beta}} + \right. \\
& \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} (g(b) - g(a))^{\beta} \left(\sup_{x \in [a,b]} \left\| D_{x-;g}^{\beta} f \right\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \left\| D_{*x;g}^{\beta} f \right\|_{\infty,[x,b]} \right) \right\} \\
& =: E_{gn}^{*}. \tag{110}
\end{aligned}$$

Hence it holds

$$\|S_n f - f\|_{\infty} \leq K(q) E_{gn}^{*}. \tag{111}$$

Next we get

$$\begin{aligned}
\left\| D_{*x;g}^{\beta} f(s) \right\| & \leq \frac{1}{\Gamma(N-\beta)} \int_x^s (g(s) - g(t))^{N-\beta-1} g'(t) \left\| (f \circ g^{-1})^{(N)}(g(t)) \right\| dt \leq \\
& \frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty, \mathbb{R}}}{\Gamma(N-\beta)} \int_{g(x)}^{g(s)} (g(s) - g(t))^{N-\beta-1} dg(t) = \\
& \frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty}}{\Gamma(N-\beta+1)} (g(s) - g(x))^{N-\beta}. \tag{112}
\end{aligned}$$

That is

$$\begin{aligned}
\left\| D_{*x;g}^{\beta} f(s) \right\| & \leq \frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty, \mathbb{R}}}{\Gamma(N-\beta+1)} (g(s) - g(x))^{N-\beta} \leq \\
& \frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty}}{\Gamma(N-\beta+1)} (g(b) - g(a))^{N-\beta}. \tag{113}
\end{aligned}$$

$\forall s \in [x, b]$.

Clearly then

$$D_{*x;g}^\beta f(x) = 0. \quad (114)$$

Acting similarly we have:

$$\begin{aligned} \|D_{x-;g}^\beta f(s)\| &\leq \frac{1}{\Gamma(N-\beta)} \int_s^x (g(t) - g(s))^{N-\beta-1} g'(t) \left\| (f \circ g^{-1})^{(N)}(g(t)) \right\| dt \leq \\ &\frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty, \mathbb{R}}}{\Gamma(N-\beta)} \int_{g(s)}^{g(x)} (g(t) - g(s))^{N-\beta-1} dg(t) = \\ &\frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty, \mathbb{R}}}{\Gamma(N-\beta+1)} (g(x) - g(s))^{N-\beta}. \end{aligned} \quad (115)$$

That is

$$\begin{aligned} \|D_{x-;g}^\beta f(s)\| &\leq \frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty, \mathbb{R}}}{\Gamma(N-\beta+1)} (g(x) - g(s))^{N-\beta} \leq \\ &\frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty}}{\Gamma(N-\beta+1)} (g(b) - g(a))^{N-\beta}. \end{aligned} \quad (116)$$

$\forall s \in [a, x]$.

Clearly then

$$D_{x-;g}^\beta f(x) = 0. \quad (117)$$

Also we see that

$$\begin{aligned} \sup_{x \in [a, b]} \|D_{*x;g}^\beta f\|_{\infty, [x, b]}, \quad \sup_{x \in [a, b]} \|D_{x-;g}^\beta f\|_{\infty, [a, x]} &\leq \\ \frac{\left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty}}{\Gamma(N-\beta+1)} (g(b) - g(a))^{N-\beta}. \end{aligned} \quad (118)$$

Furthermore it holds

$$\begin{aligned} \sup_{x \in [a, b]} \omega_1 \left(D_{x-;g}^\beta f; \frac{1}{n^\alpha} \right)_{[a, x]}, \quad \sup_{x \in [a, b]} \omega_1 \left(D_{*x;g}^\beta f; \frac{1}{n^\alpha} \right)_{[x, b]} &\leq \\ \frac{2 \left\| (f \circ g^{-1})^{(N)} \circ g \right\|_{\infty}}{\Gamma(N-\beta+1)} (g(b) - g(a))^{N-\beta}. \end{aligned} \quad (119)$$

That is $E_{gn}^* < \infty$.

We finally notice that

$$\begin{aligned} S_n(f, x) - \sum_{j=1}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} S_n \left((g(\cdot) - g(x))^j \right)(x) - f(x) &\stackrel{(35)}{=} \\ \frac{S_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k)} - \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k)} & \\ \left(\sum_{j=1}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} S_n^* \left((g(\cdot) - g(x))^j \right)(x) - f(x) \right) &= \end{aligned} \quad (120)$$

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k)} [S_n^*(f, x) - \sum_{j=1}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} S_n^*((g(\cdot) - g(x))^j)(x) - \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \right) f(x)] . \quad (121)$$

Therefore we get

$$\begin{aligned} & \left\| W_n(f, x) - \sum_{j=1}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} S_n((g(\cdot) - g(x))^j)(x) - f(x) \right\| \leq \\ & K(q) \left\| S_n^*(f, x) - \sum_{j=1}^{N-1} \frac{(f \circ g^{-1})^{(j)}(g(x))}{j!} S_n^*((g(\cdot) - g(x))^j)(x) \right. \\ & \quad \left. - \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx-k) \right) f(x) \right\|, \quad \forall x \in [a, b]. \end{aligned} \quad (122)$$

The proof of the theorem now is completed. $\square$

The case $N = 1$ follows next directly from Theorem 5.1.

Theorem 5.2. *Let $0 < \alpha, \beta < 1$, $f \in C^1([a, b], X)$, $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$ strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Also $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then*

1)

$$\begin{aligned} & \|S_n(f, x) - f(x)\| \leq \\ & \frac{K(q)}{\Gamma(\beta+1)} \inf_g \left\{ \left(\omega_1 \left(D_{x-;g}^\beta f; \frac{1}{n^\alpha} \right)_{[a,x]} + \omega_1 \left(D_{*x;g}^\beta f; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \right. \\ & \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\|D_{x-;g}^\beta f\|_{\infty,[a,x]} (g(x) - g(a))^\beta + \|D_{*x;g}^\beta f\|_{\infty,[x,b]} (g(b) - g(x))^\beta \right) \right\}, \end{aligned} \quad (123)$$

and

2)

$$\begin{aligned} & \|S_n(f) - f\|_\infty \leq \\ & \frac{K(q)}{\Gamma(\beta+1)} \inf_g \left\{ \left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-;g}^\beta f; \frac{1}{n^\alpha} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x;g}^\beta f; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\|g^{(1)}\|_\infty^\beta}{n^{\alpha\beta}} + \right. \\ & \left. (g(b) - g(a))^\beta \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\sup_{x \in [a,b]} \|D_{x-;g}^\beta f\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \|D_{*x;g}^\beta f\|_{\infty,[x,b]} \right) \right\}. \end{aligned} \quad (124)$$

Here the rate of convergence is $n^{-\alpha(\beta+1)}$.

When $\beta = \frac{1}{2}$ from Theorem 5.2 we derive:

Corollary 5.3. *Let $0 < \alpha < 1$, $f \in C^1([a, b], X)$, $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$ strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Also $x \in [a, b]$, $n \in \mathbb{N} : n > 4$. Then*

1)

$$\begin{aligned} & \|S_n(f, x) - f(x)\| \leq \\ & \frac{2K(q)}{\sqrt{\pi}} \inf_g \left\{ \left(\omega_1 \left(D_{x-;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[a,x]} + \omega_1 \left(D_{*x;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\sqrt{\|g^{(1)}\|_\infty}}{n^{\frac{\alpha}{2}}} + \right. \\ & \left. \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\left\| D_{x-;gf}^{\frac{1}{2}} \right\|_{\infty, [a,x]} \sqrt{(g(x) - g(a))} + \left\| D_{*x;gf}^{\frac{1}{2}} \right\|_{\infty, [x,b]} \sqrt{(g(b) - g(x))} \right) \right\}, \end{aligned} \quad (125)$$

and

2)

$$\begin{aligned} & \|S_n(f) - f\|_\infty \leq \\ & \frac{2K(q)}{\sqrt{\pi}} \inf_g \left\{ \left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[x,b]} \right) \frac{\sqrt{\|g^{(1)}\|_\infty}}{n^{\frac{\alpha}{2}}} + \right. \\ & \left. \sqrt{(g(b) - g(a))} \gamma A^{-\lambda(n^{1-\alpha}-2)} \left(\sup_{x \in [a,b]} \left\| D_{x-;gf}^{\frac{1}{2}} \right\|_{\infty, [a,x]} + \sup_{x \in [a,b]} \left\| D_{*x;gf}^{\frac{1}{2}} \right\|_{\infty, [x,b]} \right) \right\}. \end{aligned} \quad (126)$$

We finish with

Remark 5.1. We perform some convergence analysis: Let $0 < \alpha < 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. We work on (126). Assume that

$$\omega_1 \left(D_{x-;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[a,x]} \leq \frac{K_1^*}{n^\alpha}, \quad (127)$$

and

$$\omega_1 \left(D_{*x;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[x,b]} \leq \frac{K_2^*}{n^\alpha}, \quad (128)$$

$\forall x \in [a, b]$, $\forall n \in \mathbb{N}$, where $K_1^*, K_2^* > 0$.

Hence we have

$$\begin{aligned} & \left[\sup_{x \in [a,b]} \omega_1 \left(D_{x-;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x;gf}^{\frac{1}{2}}; \frac{1}{n^\alpha} \right)_{[x,b]} \right] \leq \\ & \frac{\frac{K_1^* + K_2^*}{n^\alpha}}{n^{\frac{\alpha}{2}}} = \frac{(K_1^* + K_2^*)}{n^{\frac{3\alpha}{2}}} = \frac{K^*}{n^{\frac{3\alpha}{2}}}, \end{aligned} \quad (129)$$

where $K^* := K_1^* + K_2^* > 0$.

The other summand of the right hand side of (126), for large enough n , converges to zero at the speed $\frac{1}{n^{1-\alpha}}$, so it is about $\frac{L^*}{n^{1-\alpha}}$, where $L^* > 0$ is a constant.

Then, for large enough $n \in \mathbb{N}$, by (126), (129) and the above comment, we obtain that

$$\|S_n(f) - f\|_\infty \leq \frac{M^*}{\min\left(n^{\frac{3\alpha}{2}}, n^{1-\alpha}\right)}, \quad (130)$$

where $M^* > 0$.

Clearly we distinguish two cases:

i)

$$\|S_n(f) - f\|_\infty \leq \frac{M^*}{n^{1-\alpha}}, \text{ where } \frac{2}{5} \leq \alpha < 1, \quad (131)$$

with speed of convergence $\frac{1}{n^{1-\alpha}}$,

and

ii)

$$\|S_n(f) - f\|_\infty \leq \frac{M^*}{n^{\frac{3\alpha}{2}}}, \text{ where } 0 < \alpha < \frac{2}{5}, \quad (132)$$

with speed of convergence $\frac{1}{n^{\frac{3\alpha}{2}}}$.

In Theorem 2.3, for $f \in C([a, b], X)$ and for large enough $n \in \mathbb{N}$, when $0 < \alpha \leq \frac{1}{2}$, the speed is $\frac{1}{n^\alpha}$. So when $0 < \alpha < \frac{2}{5}$ ($< \frac{1}{2}$), we get by (132) that $\|S_n(f) - f\|_\infty$ converges much faster to zero. The last comes because we assumed differentiability of f .

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Received 30 June 2026

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